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On Sums of Fractional Parts $\{n\alpha+\gamma\}$

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$$C_m(\alpha,\gamma)=\sum_{1\leqslant k\leqslant m}\left(\{k\alpha+\gamma\}-\tfrac{1}{2}\right)$$

when α is irrational. From this we deduce a number of elementary bounds on the growth and behaviour of $C_m(\alpha,\gamma)$. In particular, we show that as m varies the extent of the fluctuations in size can be determined almost entirely from the non-homogeneous continued fraction expansion of γ with respect to α. These sums are closely related to the discrepancy of the sequence $(\{n\alpha\})$; we state a related explicit formula that yields similar bounds for the discrepancy. Sums of this form also occur in a lattice point problem of Hardy and Littlewood. © 1997 Academic Press

1. INTRODUCTION

In a recent paper, Brown and Shiue [3] use the continued fraction expansion of α to obtain an explicit formula for the sum

$$C_\alpha(m)=\sum_{1\leqslant k\leqslant m}\left(\{k\alpha\}-\tfrac{1}{2}\right)$$

and to give simple proofs of results of Lerch [12], Hardy and Littlewood [8; 9, "Problem B"], Ostrowski [14], and Sós [20]. In particular, in addition to producing explicit upper and lower bounds for $|C_\alpha(m)|$, they show that if the partial quotients of α are bounded by A then, for some explicit constant d_A, both $C_\alpha(m)>d_A\log m$ and $C_\alpha(m)<-d_A\log m$ hold for infinitely many m.

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We show here that a surprisingly similar formula holds in the non-homogeneous case,

$$C_m(\alpha, \gamma) = \sum_{1 \leqslant k \leqslant m} (\{k\alpha + \gamma\} - \tfrac{1}{2}),$$

leading to comparable upper and lower bounds for the absolute value of this quantity. There is at least one major difference though; it is no longer true that bounded quotients are sufficient to cause the sizeable positive–negative swings that occur when $\gamma = 0$. For example (as we shall show in a subsequent paper), if $\alpha = \sqrt{2}$ one-sidedly bounded sums $C_m(\sqrt{2}, \frac{1}{2}) > 0$ occur when $\gamma = 1/2$. However, such a gamma should be thought of as exceptional. We show that the extent or absence of these fluctuations is (in some asymptotic sense) determined by the non-homogeneous continued fraction expansion of γ with respect to α.

Sums of the form $C_{m/\omega}(\alpha, -\alpha(m/\omega))$ were studied in detail by Hardy and Littlewood [8; 9, "Problem A"] in connection with the problem of approximating the number of lattice points in a right-angled triangle. Most of their bounds follow straightforwardly from ours. The sums $C_m(\alpha, \gamma)$ are also closely related to the discrepancy of the sequence $(\{n\alpha\})$; in particular, our formula yields expressions reminiscent of the "explicit formulae" of Sós and Dupain [5–7, 21–23] and could be used to duplicate many of their results. Similar formulae and estimates appear in the work of Schoissengeier [1, 16–19].

In the next section we introduce the various notations and give the machinery and basic properties of the regular and non-homogeneous continued fraction expansions; we postpone the proof of those propositions until the end of Section 5. In Section 3 we state the main results; we give the proofs in Section 5. In Section 4 we state without proof the corresponding formula and results for the discrepancy function.

2. BASIC NOTATION

For any irrational real α, real γ and integer $m \geqslant 1$, we write

$$C_m(\alpha, \gamma) := \sum_{1 \leqslant k \leqslant m} (\{k\alpha + \gamma\} - \tfrac{1}{2})$$

where as usual $\{x\} = x - [x]$ denotes the fractional part of x.

We shall suppose throughout that α is irrational and has the continued fraction expansion

$$\alpha = [a_0, a_1, a_2, ...],$$

where

$$a_{n+1} := \left[\frac{1}{\alpha_n}\right], \qquad \alpha_{n+1} := \left\{\frac{1}{\alpha_n}\right\}, \qquad \alpha_0 := \{\alpha\}, \tag{1}$$

and shall use p_i/q_i to denote the ith convergent to α,

$$p_{n+1} := a_{n+1} p_n + p_{n-1}, \qquad p_{-1} := 1, \quad p_{-2} := 0,$$
$$q_{n+1} := a_{n+1} q_n + q_{n-1}, \qquad q_{-1} := 0, \quad q_{-2} := 1,$$

with

$$\varepsilon_i := q_i \alpha - p_i = \frac{(-1)^i}{q_{i+1} + \alpha_{i+1} q_i}$$

denoting the closeness of such an approximation.

Following Brown–Shiue, we shall make frequent use of the unique decomposition of an $m < q_t$ as

$$m = z_t q_{t-1} + \cdots + z_2 q_1 + z_1 q_0$$

(the so called "Zeckendorff Representation" of m), where

(i) $0 \leqslant z_1 \leqslant a_1 - 1$,
(ii) $0 \leqslant z_i \leqslant a_i$, $2 \leqslant i \leqslant t$,
(iii) if $z_i = a_i$ then $z_{i-1} = 0$ $(2 \leqslant i \leqslant t)$,

and use m_j, $1 \leqslant j \leqslant t$, to denote the corresponding subsums

$$m_j = z_1 q_0 + \cdots + z_j q_{j-1}.$$

Note that for all j

$$m_j + m_{j-1} + 1 \leqslant q_j. \tag{2}$$

We shall also need some new gamma dependent parameters

$$\beta_n = \beta_n(\alpha, \gamma) := \begin{cases} \{\gamma q_{n-1}\} & \text{if } n \text{ is even}, \\ 1 - \{\gamma q_{n-1}\} & \text{if } n \text{ is odd}, \end{cases} \tag{3}$$

and

$$u_n = u_n(\alpha, \gamma) := \min\left\{k \in N : [k\alpha + \gamma] \neq \left[k \frac{p_{n-1}}{q_{n-1}} + \gamma\right]\right\}. \tag{4}$$

The following proposition enables us to explicitly compute the u_i:

PROPOSITION 1.

$$u_n = \beta_n q_n + (\lambda_n + \beta_{n+1})\, q_{n-1}$$

where

$$\lambda_n := \begin{cases} 1 & \text{if } \alpha_n\beta_n > \beta_{n+1} \text{ (or } \alpha_n\beta_n = \beta_{n+1} \text{ if } n \text{ is even)}, \\ 0 & \text{if } \alpha_n\beta_n < \beta_{n+1} \text{ (or } \alpha_n\beta_n = \beta_{n+1} \text{ if } n \text{ is odd)}. \end{cases}$$

We shall also need to consider the non-homogeneous continued fraction expansion of γ with respect to α (see Borwein–Borwein [2] for more details). Suppose that the continued fraction for α produces a sequence of a_n and α_n as above, then we generate an accompanying sequence of non-homogeneous partial quotients c_n and remainders γ_n by setting

$$c_{n+1} := \left[\frac{\gamma_n}{\alpha_n}\right], \qquad \gamma_{n+1} := \left\{\frac{\gamma_n}{\alpha_n}\right\}, \qquad \gamma_0 := \{\gamma\}. \tag{5}$$

Noting the relation

$$\{\gamma\} = \sum_{i=1}^{n} c_i\, |\varepsilon_{i-1}| + \gamma_n\, |\varepsilon_{n-1}| \tag{6}$$

the c_i give us an expansion of γ in terms of α:

PROPOSITION 2. *For* $0 \leqslant \gamma < 1$,

$$\gamma = \sum_{i=1}^{\infty} c_i\, |\varepsilon_{i-1}|,$$

where the c_i *produced by* (5) *have the following properties*:

(i) $0 \leqslant c_i \leqslant a_i$

(ii) *if* $c_i = a_i$ *then* $c_{i+1} = 0$

(iii) $c_i \neq a_i$ *for infinitely many odd and infinitely many even i.*

Moreover, such an expansion is unique (*i.e. if* $\gamma = \sum_{i=1}^{\infty} b_i\, |\varepsilon_{i-1}|$ *with integers* b_i *satisfying* (i), (ii), *and* (iii) *then* $b_i = c_i$ *for all i*).

Note that this expansion is distinct from that employed by Sós and Dupain [5, 21–23] (attributed by them to Lesca [13] and Descombes [4]) where ε_{i-1} replace the $|\varepsilon_{i-1}|$.

We set v_n to be the sums

$$v_n := \sum_{i=1}^{n} (-1)^{n-i} c_i q_{i-1}$$

and observe that one can express the β_n and u_n in terms the v_i and γ_i.

PROPOSITION 3.

$$\beta_n = l_n + (-1)^n (v_n + \gamma_n q_{n-1}) \, |\varepsilon_{n-1}|$$
$$u_n = l_n q_n + (l_{n+1} + \lambda_n) \, q_{n-1} + (-1)^n \, v_n$$

where

$$l_n := \begin{cases} 1 & \textit{if } (-1)^n (v_n + \gamma_n q_{n-1}) < 0 \quad (\textit{or } = \textit{if } n \textit{ is odd}), \\ 0 & \textit{if } (-1)^n (v_n + \gamma_n q_{n-1}) > 0 \quad (\textit{or } = \textit{if } n \textit{ is even}), \end{cases}$$

$$\lambda_n := \begin{cases} 1 & \textit{if } \alpha_n l_n + (-1)^n \gamma_n > l_{n+1} \quad (\textit{or } = \textit{if } n \textit{ is even}), \\ 0 & \textit{if } \alpha_n l_n + (-1)^n \gamma_n < l_{n+1} \quad (\textit{or } = \textit{if } n \textit{ is odd}). \end{cases}$$

The parameters v_n appear in Borwein–Borwein [2] disguised as $t_{n-1} = q_n + q_{n-1} + (-1)^n v_n$. We note the elementary bounds

$$-q_{n-1} \leqslant v_n \leqslant q_n.$$

Proposition 3 is perhaps more digestible in its expanded form:

When n is odd,

$$u_n = \begin{cases} q_n + q_{n-1} - v_n & \text{if} \quad c_n \neq 0 \text{ (or } c_n = 0,\ \gamma_n q_{n-1} \geqslant v_{n-1}) \text{ and } c_{n+1} = 0, \\ q_n - v_n & \text{if} \quad c_n \neq 0 \text{ (or } c_n = 0,\ \gamma_n q_{n-1} \geqslant v_{n-1}) \text{ and } c_{n+1} \neq 0, \\ v_{n-1} & \text{if} \quad c_n = 0 \text{ and } \gamma_n q_{n-1} < v_{n-1}, \end{cases}$$

and when n is even,

$$u_n = \begin{cases} q_{n-1} + v_n & \text{if} \quad c_n \neq 0 \text{ (or } c_n = 0,\ \gamma_n q_{n-1} \geqslant v_{n-1}), \\ q_n + q_{n-1} - v_{n-1} & \text{if} \quad c_n = 0 \text{ and } \gamma_n q_{n-1} < v_{n-1} \text{ and } \alpha_n + \gamma_n < 1, \\ q_n + 2q_{n-1} - v_{n-1} & \text{if} \quad c_n = 0 \text{ and } \gamma_n q_{n-1} < v_{n-1} \text{ and } \alpha_n + \gamma_n \geqslant 1. \end{cases}$$

In several of the proofs we shall make use of the parameters

$$\delta_n = \delta_n(\alpha, \gamma) := \{\gamma q_{n-1}\}, \qquad d_n = d_n(\alpha, \gamma) := [\gamma q_{n-1}]. \tag{7}$$

We also recall the common notations

$$x^+ = \max(x, 0), \qquad x^- = \min(x, 0)$$

and $\|x\|$, the distance from x to the nearest integer. Finally, we define a useful variant of the integer part

$$[x]_* = \begin{cases} [x] & \text{if } \ x \notin \mathbb{Z}, \\ x-1 & \text{if } \ x \in \mathbb{Z}. \end{cases} \tag{8}$$

3. OUR MAIN RESULTS

With β_i, u_i and $[x]_*$ defined in (3), (4) and (8) above, we show the following simple, explicit, formula for the sum $C_m(\alpha, \gamma)$:

THEOREM 1. *If* $m = z_1 q_0 + \cdots z_t q_{t-1}$ *is the Zeckendorff representation of* $m \geqslant 1$, *then*

$$C_m(\alpha, \gamma) = \sum_{1 \leqslant i \leqslant t} (-1)^i M_i,$$

where

$$M_i = -\frac{1}{2} z_i \, |\varepsilon_{i-1}| \, (m_i + m_{i-1} + 1) + \left(\beta_i - \frac{1}{2}\right) z_i + \left(z_i - \left[\frac{u_i - m_{i-1}}{q_{i-1}}\right]_*^+\right)^+.$$

Note that when $\gamma = 0$,

$$\beta_i = \begin{cases} 0 & \text{if } i \text{ is even}, \\ 1 & \text{if } i \text{ is odd}, \end{cases} \qquad u_i = \begin{cases} q_{i-1} & \text{if } i \text{ is even}, \\ q_i + q_{i-1} & \text{if } i \text{ is odd}, \end{cases}$$

giving (for $z_i \leqslant a_i$)

$$\left(z_i - \left[\frac{u_i - m_{i-1}}{q_{i-1}}\right]_*^+\right)^+ = \begin{cases} z_i & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}, \end{cases}$$

and we immediately recover Brown–Shiue [3, Theorem 1(c)],

$$C_m(\alpha, 0) = \sum_{i=1}^{t} (-1)^i \tfrac{1}{2} z_i (1 - |\varepsilon_{i-1}| \, (m_i + m_{i-1} + 1)). \tag{9}$$

Rough estimation readily gives us a rough upper bound:

COROLLARY 1. *With m as above*

$$|C_m(\alpha, \gamma)| \leqslant \tfrac{3}{2} \sum_{i=1}^{t} z_i, \qquad \max_{1 \leqslant m < q_t} |C_m(\alpha, \gamma)| < \tfrac{3}{2} \sum_{i=1}^{t} a_i.$$

With a bit more effort one can show more precisely how the partial quotients a_i of α and the non-homogeneous partial quotients c_i of γ with respect to α (as defined in (5) above), affect the growth of $C_m(\alpha, \gamma)$:

COROLLARY 2. (i) *If we fix* α *and* γ *and vary* m *then, for any* $t \geqslant 1$,

$$\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = \frac{1}{2} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} \frac{c_i}{a_i}\left(1 - \frac{c_i}{a_i}\right) a_i + \frac{1}{2} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} \left(\frac{1}{2} - \frac{c_i}{a_i}\right)^2 a_i + E_1(t),$$

$$\min_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = -\frac{1}{2} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} \frac{c_i}{a_i}\left(1 - \frac{c_i}{a_i}\right) a_i - \frac{1}{2} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} \left(\frac{1}{2} - \frac{c_i}{a_i}\right)^2 a_i - E_2(t),$$

where $|E_i(t)| \leqslant \frac{1}{2}(5t+1)$.

(ii) *If we fix* α *and* $m = z_1 q_0 + \cdots + z_t q_{t-1}$ *and vary* γ, *then*

$$\sup_{\gamma \in [0,1)} C_m(\alpha, \gamma) = \frac{1}{2} \sum_{i=1}^{t} \frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i + F_1(t),$$

$$\inf_{\gamma \in [0,1)} C_m(\alpha, \gamma) = -\frac{1}{2} \sum_{i=1}^{t} \frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i - F_2(t),$$

where $|F_i(t)| \leqslant \frac{1}{2}(5t+1)$.

Similarly, if we vary both m and γ,

$$\sup_{\gamma \in [0,1)} \max_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = \tfrac{1}{8} \sum_{i=1}^{t} a_i + G_1(t),$$

$$\inf_{\gamma \in [0,1)} \min_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = -\tfrac{1}{8} \sum_{i=1}^{t} a_i - G_2(t),$$

with $|G_i(t)| \leqslant \frac{1}{2}(5t+1)$.

We observe that the right-hand sides of the expressions in (i) are attained for the choice $m'_\gamma := \sum z'_i\, q_{i-1}$ and $m''_\gamma := \sum z''_i\, q_{i-1}$ respectively, where

$$z'_i := \begin{cases} [c_i \pm \frac{1}{2} a_i] & \text{if } i \text{ is even}, \\ a_i - c_i & i \text{ odd}, c_i \neq 0, \\ 0 & i \text{ odd}, c_i = 0, \end{cases} \qquad z''_i := \begin{cases} [(a_i - c_i) \pm \frac{1}{2} a_i] & \text{if } i \text{ is odd}, \\ c_i & i \text{ even}, c_i \neq a_i, \\ 0 & i \text{ even}, c_i = a_i, \end{cases}$$

with the $\pm$ sign chosen such that $0 \leqslant z'_i, z''_i < a_i$. Similarly, the right-hand sides in (ii) are achieved when

$$\gamma'_m := 1 - \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} z_i\, |\varepsilon_{i-1}|, \qquad \gamma''_m := \sum_{\substack{i=1 \\ i \text{ even}}}^{t} z_i\, |\varepsilon_{i-1}|. \tag{10}$$

Using $m^{(j)}$ to denote the sum of the odd indexed, $\sum_{i \text{ odd}} z_i q_{i-1}$, or even indexed, $\sum_{i \text{ even}} z_i q_{i-1}$, terms of the Zeckendorff representation of $m = \sum z_i q_{i-1}$, as j is odd or even respectively, we note that $\gamma'_m = \{-m^{(1)}\alpha\}$ and $\gamma''_m = \{-m^{(0)}\alpha\}$. In particular $C_m(\alpha, \gamma)$ can in these cases be rewritten as a sum of wholly positive or wholly negative terms;

$$\begin{aligned} C_m(\alpha, \gamma'_m) &= -\{\gamma'_m\} + \sum_{i=1}^{t} \tfrac{1}{2} z_i (1 - |\varepsilon_{i-1}| \, (m_i^{(i)} + m_{i-1}^{(i)} + 1)), \\ C_m(\alpha, \gamma''_m) &= -\{\gamma''_m\} - \sum_{i=1}^{t} \tfrac{1}{2} z_i (1 - |\varepsilon_{i-1}| \, (m_i^{(i)} + m_{i-1}^{(i)} + 1)), \end{aligned} \tag{11}$$

via the simple relation $C_{m+n}(\alpha, -n\alpha) = C_m(\alpha, 0) - C_n(\alpha, 0) - \{-n\alpha\}$.

We remark that for general γ it is no longer true that bounded partial quotients are sufficient to cause $C_m(\alpha, \gamma)$ to take arbitrarily large positive and negative values (recall, Hardy & Littlewood [8, Theorem B4], that if the $a_i \leqslant A$ then $C_m(\alpha, 0) > c_A \log m$ and $C_m(\alpha, 0) < -c_A \log m$ must both hold for infinitely many m). We note the values $\gamma_0 := \sum_{i=1}^{\infty} [a_{2i}/2] \, |\varepsilon_{2i-1}|$ and $\gamma_1 := \sum_{i=1}^{\infty} [a_{2i-1}/2] \, |\varepsilon_{2i-2}|$ for which the sums are particularly one-sided (notice that if the a_i are all even then $\gamma_0 = \frac{1}{2}$ and $\gamma_1 = \frac{1}{2}\{\alpha\}$)

$$\begin{aligned} &\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma_0) = \tfrac{1}{8} \sum_{i=1}^{t} a_i + O(t), \qquad &&\min_{1 \leqslant m < q_t} C_m(\alpha, \gamma_0) = O(t), \\ &\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma_1) = O(t), &&\min_{1 \leqslant m < q_t} C_m(\alpha, \gamma_1) = -\tfrac{1}{8} \sum_{i=1}^{t} a_i + O(t), \end{aligned} \tag{12}$$

and $\gamma_2 := \sum_{i=1}^{\infty} [(\frac{1}{2} \pm \sqrt{2}/4) \, a_i] \, |\varepsilon_{i-1}|$ for which the positive–negative swings are the most symmetric

$$\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma_2) = \tfrac{1}{16} \sum_{i=1}^{t} a_i + O(t), \qquad \min_{1 \leqslant m < q_t} C_m(\alpha, \gamma_2) = -\tfrac{1}{16} \sum_{i=1}^{t} a_i + O(t). \tag{13}$$

For $\gamma = 0$ the corollary gives (similar to Schoissengeier [18])

$$\max_{1 \leqslant m < q_t} C_m(\alpha, 0) = \tfrac{1}{8} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} a_i + O(t), \qquad \min_{1 \leqslant m < q_t} C_m(\alpha, 0) = -\tfrac{1}{8} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} a_i + O(t).$$

In contrast, the difference between the largest and smallest value is little affected by the choice of γ:

$$\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma) - \min_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = \tfrac{1}{8} \sum_{i=1}^{t} a_i + O(t).$$

Using the parameters β_i of (3) rather than the c_i, we can similarly obtain the less discrete but perhaps more straightforward variant of (i),

$$\max_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = \tfrac{1}{2} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} \beta_i(1-\beta_i)\, a_i + \tfrac{1}{2} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} (\tfrac{1}{2}-\beta_i)^2\, a_i + E_3(t),$$

$$\min_{1 \leqslant m < q_t} C_m(\alpha, \gamma) = -\tfrac{1}{2} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} \beta_i(1-\beta_i)\, a_i - \tfrac{1}{2} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} (\tfrac{1}{2}-\beta_i)^2\, a_i - E_4(t),$$

where $-\frac{1}{4}(11t+1) \leqslant E_i(t) \leqslant \frac{1}{2}(5t+1)$. The connection becomes clear on observing that $\{\gamma q_{i-1}\} = c_i/a_i + O(1/a_i)$ if $c_i \neq 0$, with $\{\gamma q_{i-1}\} = O(1/a_i)$ or $1 - O(1/a_i)$ if $c_i = 0$.

As an easy consequence of Corollary 2 we have the following upper and lower bounds on the growth rate of $|C_m(\alpha, \gamma)|$:

Corollary 3. *For* $t \geqslant 1$,

$$\tfrac{1}{16} \sum_{i=1}^{t} (a_i - 40)^+ \leqslant \max_{1 \leqslant m < q_t} |C_m(\alpha, \gamma)| \leqslant \tfrac{1}{8} \sum_{i=1}^{t} (a_i + 24).$$

In view of (12) and (13) the constants 1/8 and 1/16 are plainly optimal. Using Corollary 3, a number of well known properties of $C_m(\alpha, 0)$ extend immediately to our more general sums $C_m(\alpha, \gamma)$. We should remark that similar upper bounds (although without our explicit constants) could be alternatively obtained from known results for the discrepancy via the relations (14) and (15) below.

Property 1. (a) If $\sum_{i=1}^{t} a_i \leqslant At$ for all $t \geqslant 1$, then

$$|C_m(\alpha, \gamma)| < \tfrac{1}{3}(A+24) \log(3m)$$

for all γ and $m \geqslant 1$.

(b) If $\sum_{i=1}^{t} a_i \geqslant (B+40)t$ for infinitely many t then, for any fixed γ,

$$|C_m(\alpha, \gamma)| > \frac{1}{16} \frac{B}{(B+40)} \log m$$

for infinitely many m.

Under the stronger hypothesis $a_i \leqslant A$ for all i in (a), or $a_i \geqslant B$ for all i in (b), one can replace the corresponding bound by

$$|C_m(\alpha, \gamma)| \leqslant \frac{1}{8} \left(\frac{A}{\log A} + 55 \right) \log(3m), \qquad |C_m(\alpha, \gamma)| \geqslant \frac{1}{16} \frac{(B-9)}{\log(B+1)} \log m,$$

respectively (a bound which is then asymptotically sharp in terms of A or B).

More generally, for a fixed α the sums $|C_m(\alpha, \gamma)|$ are $o(m)$ as $m \to \infty$ (uniformly in γ) but are not $o(m)$ uniformly in α:

PROPERTY 2. (a) For any fixed α and all γ

$$|C_m(\alpha, \gamma)| \leqslant 3 \frac{m}{\max\{q_s < \sqrt{m}\}} = o_\alpha(m)$$

as $m \to \infty$.

(b) For any $t \geqslant 1$

$$\max_{m < q_t} |C_m(\alpha, \gamma)| \geqslant \frac{1}{16} \left(\frac{q_t}{q_{t-1}} - 41 \right).$$

In particular, given any function $f(n) = o(n)$, there are infinitely many α such that, for any fixed γ,

$$\limsup_{n \to \infty} \left| \frac{C_n(\alpha, \gamma)}{f(n)} \right| = \infty.$$

In general a precise knowledge of the growth of the partial quotients of α (equivalently the quality of rational approximations to α) leads to accurate bounds on the growth of $|C_m(\alpha, \gamma)|$. We give the following primarily to show that most of the results of Hardy–Littlewood [9, Theorems 2, 3] do still hold for these more general sums (similar theorems occur in Ostrowski [14, pp. 80–81]).

PROPERTY 3. (a) For any $r \geqslant 0$ and non-decreasing function f, such that $q^{1+r} f(q) \|q\alpha\| > 1$ for all $q \in \mathbb{N}$,

$$|C_m(\alpha, \gamma)| < 4m^{r/1+r} f(m)^{1/1+r} \log(3m)$$

for all γ and $m \geqslant 1$.

(b) If, for some fixed $r > 0$, $q^{1+r} \|q\alpha\| < 1$ for infinitely many $q \in \mathbb{N}$ then, for any fixed γ,

$$|C_m(\alpha, \gamma)| > \tfrac{1}{64} m^{r/1+r}$$

for infinitely many m.

Notice from (a) that if α is algebraic then, by Roth's Theorem [15], for any $\varepsilon > 0$ there is a constant $c_1(\alpha, \varepsilon)$ such that $|C_m(\alpha, \gamma)| < c_1(\alpha, \varepsilon)\, m^{\varepsilon}$.

4. THE DISCREPANCY OF THE SEQUENCE $(\{n\alpha\})$

For a sequence $\mathscr{S}=(b_i)$, $b_i \in [0, 1)$ one measures how close the subinterval I of $[0, 1)$ comes to receving its "fair share" of points by means of the function:

$$\Delta_N(I, \mathscr{S}) := \sum_{i=1}^{N} (\chi_I(b_i) - |I|),$$

where $\chi_I(x)$ denotes the characteristic function of I and $|I|$ the length of I. We recall the definition of the *discrepancy* $D_N(\mathscr{S}) := \sup_I |\Delta_N(I, \mathscr{S})|$ of $\mathscr{S}$ and its variant, the *extreme discrepancy* $D_N^*(\mathscr{S})$ of $\mathscr{S}$, that we shall use here

$$D_N^*(\mathscr{S}) := \sup_{\beta} |\Delta_N([0, \beta), \mathscr{S})|.$$

For the sequences $\mathscr{S}=(\{n\alpha\})_{n=1}^{\infty}$ we use the abbreviations

$$\Delta_N(\beta, \alpha) := \Delta_N([0, \beta), (\{n\alpha\})_{n=1}^{\infty}), \qquad D_N^*(\alpha) := ((\{n\alpha\})_{n=1}^{\infty}).$$

Since for $0 \leqslant \gamma < 1$

$$\{n\alpha + \gamma\} = \{n\alpha\} + \chi_{[0, 1-\gamma)}(\{n\alpha\}) - (1-\gamma)$$

we observe the following simple relation between the $C_m(\alpha, \gamma)$ and $\Delta_N(\beta, \alpha)$;

$$C_m(\alpha, \gamma) = C_m(\alpha, 0) + \Delta_m(1-\gamma, \alpha). \tag{14}$$

Lesca [3] has shown further that

$$C_m(\alpha, 0) = -\tfrac{1}{2}\Delta_m(\{(m-1)\,\alpha\}, \alpha), \tag{15}$$

so that the discrepancy formulae of Sós et al. could presumably be conversely used to obtain a related formula for $C_m(\alpha, \gamma)$. From (14) and Theorem 1 we obtain at once the following explicit formula for the discrepancy:

Corollary 4. *If $m = z_1 q_0 + \cdots + z_t q_{t-1}$ is the Zeckendorff representation of $m \geqslant 1$ and $0 \leqslant \gamma < 1$, then*

$$\Delta_m(1-\gamma, \alpha) = \sum_{i=1}^{t} (-1)^i H_i$$

where

$$H_i = -(1-\beta_i)\, z_i + \left(z_i - \left[\frac{u_i - m_{i-1}}{q_{i-1}}\right]_*^+\right)^+.$$

We immediately obtain the rough bounds

$$D_m^*(\alpha) \leqslant \sum_{i=1}^{t} z_i, \qquad \max_{1 \leqslant m < q_t} D_m^*(\alpha) < \sum_{i=1}^{t} a_i.$$

Corresponding to Corollary 2 we observe the asymptotically precise bounds

COROLLARY 5. (i) *If we fix* α and γ *and vary* m *then, for any* $t \geqslant 1$,

$$\max_{1 \leqslant m < q_t} \Delta_m(1-\gamma, \alpha) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} \frac{c_i}{a_i}\left(1 - \frac{c_i}{a_i}\right) a_i + E_1(t),$$

$$\min_{1 \leqslant m < q_t} \Delta_m(1-\gamma, \alpha) = -\sum_{\substack{i=1 \\ i \text{ even}}}^{t} \frac{c_i}{a_i}\left(1 - \frac{c_i}{a_i}\right) a_i - E_2(t),$$

where $-(t+1) \leqslant E_i \leqslant \frac{1}{2}(5t+1)$.

(ii) *If we fix* α *and* $m = z_1 q_0 + \cdots + z_t q_{t-1}$ *and vary* γ *then*

$$\sup_{\gamma \in [0,1)} \Delta_m(1-\gamma, \alpha) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} \frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i + F_1(t),$$

$$\inf_{\gamma \in [0,1)} \Delta_m(1-\gamma, \alpha) = -\sum_{\substack{i=1 \\ i \text{ even}}}^{t} \frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i - F_2(t),$$

where $-(t+1) \leqslant F_i \leqslant \frac{1}{2}(5t+1)$.

It is perhaps worth recalling here the theorem of Kesten [10]; namely that if the partial quotients of α are bounded, then $\Delta_m(1-\gamma, \alpha)$ is bounded if and only if $\gamma = \{n\alpha\}$ for some integer n (equivalently $c_i = 0$ for all but finitely many i). Notice that, varying both m and γ,

$$\sup_{\gamma \in [0,1)} \max_{1 \leqslant m < q_t} \Delta_m(\gamma, \alpha) = \frac{1}{4} \sum_{\substack{i=1 \\ i \text{ odd}}}^{t} a_i + G_1(t),$$

$$\inf_{\gamma \in [0,1)} \min_{1 \leqslant m < q_t} \Delta_m(\gamma, \alpha) = -\frac{1}{4} \sum_{\substack{i=1 \\ i \text{ even}}}^{t} a_i - G_2(t),$$

where $-\frac{9}{8}(t+1) \leqslant G_i(t) \leqslant \frac{1}{2}(5t+1)$. Expressions similar to this and (ii) appear in Schoissengeier and Baxa [1, 16, 17, 19]. The right-hand sides in (i) are in this case attained for $\tilde{m}'_\gamma := \sum_{i \text{ odd}} (a_i - c_i)\, q_{i-1}$ and $\tilde{m}''_\gamma := \sum_{i \text{ even}} c_i q_{i-1}$ respectively, those in (ii) are again achieved for the γ'_m and γ''_m of (10).

Plainly there is a Corollary-3-type inequality

$$\frac{1}{8} \sum_{i=1}^{t} (a_i - 9)^+ \leqslant \max_{1 \leqslant m < q_t} D^*_m(\alpha) \leqslant \frac{1}{4} \sum_{i=1}^{t} (a_i + 12).$$

The various properties given for $C_m(\alpha, \gamma)$ likewise hold for $D^*_m(\alpha)$ after appropriate adjustments to the precise constants. Many similar results on the discrepancy can be found in Kuipers–Niederreiter [11, Chapter 3] and Sós [22, 23].

Finally we show very simply that when the partial quotients are (on average) bounded, and γ allowed to vary, $\Delta_m(\gamma, \alpha)$ must take logarithmically large and small values:

Corollary 6. *If $\sum_{i=1}^{t} a_i \leqslant At$ for infinitely many t then*

$$\sup_\gamma \Delta_m(\gamma, \alpha) > c_A \log m, \qquad \inf_\gamma \Delta_m(\gamma, \alpha) < -c_A \log m$$

each hold for infinitely many m, where we may take $c_A = 1/90A^2$.

5. THE PROOFS

We shall need the following simple, yet crucial, lemma:

Lemma 1. *For $1 \leqslant n \leqslant q_i$,*

$$[n\alpha + \gamma] \neq \left[n \frac{p_{i-1}}{q_{i-1}} + \gamma \right]$$

if and only if $n = u_i(\alpha, \gamma) + lq_{i-1}$ for some integer $l \geqslant 0$. Moreover, the difference is at most 1.

Proof. Let

$$S_i = \left\{ n_1 : \left[n_1 \frac{p_{i-1}}{q_{i-1}} + \gamma \right] \neq [n_1 \alpha + \gamma] \right\}.$$

Then if $n_1 \in S_i$,

$$[n_1\alpha + \gamma] < \left[n_1 \frac{p_{i-1}}{q_{i-1}} + \gamma \right] i \text{ even}, \qquad [n_1\alpha + \gamma] > \left[n_1 \frac{p_{i-1}}{q_{i-1}} + \gamma \right] i \text{ odd}.$$

Hence for any $n_2 = n_1 + lq_{i-1}$, $l \geqslant 1$ we have

$$\left[n_2 \frac{p_{i-1}}{q_{i-1}} + \gamma\right] = \left[n_1 \frac{p_{i-1}}{q_{i-1}} + \gamma\right] + lp_{i-1}$$

$$[n_2\alpha + \gamma] = [(n_1\alpha + \gamma) + (-1)^{i-1} l\, |\varepsilon_{i-1}|] + lp_{i-1}$$

giving

$$[n_2\alpha + \gamma] \leqslant [n_1\alpha + \gamma] + lp_{i-1} < \left[n_2 \frac{p_{i-1}}{q_{i-1}} + \gamma\right] \qquad \text{if } i \text{ is even,}$$

$$[n_2\alpha + \gamma] \geqslant [n_1\alpha + \gamma] + lp_{i-1} > \left[n_2 \frac{p_{i-1}}{q_{i-1}} + \gamma\right] \qquad \text{if } i \text{ is odd,}$$

and $n_2 \in S_i$. In particular $n \in S_i$ for any n of the form $u_i + lq_{i-1}$, $l \geqslant 0$.

Conversely, suppose n_1, n_2 are both in S_i with $n_1, n_2 \leqslant q_i$. Then for some integers m_1, m_2,

$$\begin{array}{ccc} n_1\alpha < m_1 - \gamma \leqslant n_1 \dfrac{p_{i-1}}{q_{i-1}} & & n_1\alpha \geqslant m_1 - \gamma > n_1 \dfrac{p_{i-1}}{q_{i-1}} \\ & \text{or} & \\ n_2\alpha < m_2 - \gamma \leqslant n_2 \dfrac{p_{i-1}}{q_{i-1}} & & n_2\alpha \geqslant m_2 - \gamma > n_2 \dfrac{p_{i-1}}{q_{i-1}}, \end{array}$$

as i is even or odd respectively. Subtracting and multiplying by q_{i-1} then gives

$$|(m_1 - m_2)\, q_{i-1} - (n_1 - n_2)\, p_{i-1}| < \max(n_1, n_2)\, |\varepsilon_{i-1}|.$$

Now if both n_1 and $n_2 \leqslant q_i < |\varepsilon_{i-1}|^{-1}$, integrality forces

$$(m_1 - m_2)\, q_{i-1} = (n_1 - n_2)\, p_{i-1}$$

and, by the coprimeness of p_{i-1} and q_{i-1},

$$n_1 \equiv n_2 \qquad (\text{mod } q_{i-1}).$$

In particular, any $n \leqslant q_i$ in S_i would have to be the form $u_i + lq_{i-1}$, where u_i is the smallest element of S_i.

Since

$$\left|(n\alpha + \gamma) - \left(n \frac{p_{i-1}}{q_{i-1}} + \gamma\right)\right| = n \frac{|\varepsilon_{i-1}|}{q_{i-1}} < \frac{n}{q_i q_{i-1}},$$

the difference is plainly at most 1 for all $1 \leqslant n \leqslant q_i q_{i-1}$. ∎

Proof of Theorem 1. We first analyse the related sum

$$S(m)=S(\alpha,\gamma,m):=\sum_{1\leqslant n\leqslant m}[n\alpha+\gamma]$$

and note that for any j and $m\leqslant q_{j+1}$ we can (by the above lemma) replace the α by its approximation p_j/q_j,

$$S_j(m)=S_j(\alpha,\gamma,m):=\sum_{1\leqslant n\leqslant m}\left[n\frac{p_j}{q_j}+\gamma\right],$$

at the price of a simple additional term

$$\begin{aligned}S(m)-S_j(m)&=(-1)^j\,\#\{1\leqslant n\leqslant m\colon n=u_{j+1}+lq_j,\, l\geqslant 0\}\\&=(-1)^j\left[\frac{m-u_{j+1}}{q_j}+1\right]^+.\end{aligned}$$

Hence if $m=bq_j+l<q_{j+1}$ with $0\leqslant l<q_j$,

$$\begin{aligned}S(m)&=S_j(bq_j)+\sum_{bq_j<n\leqslant bq_j+l}\left[n\frac{p_j}{q_j}+\gamma\right]+(-1)^j\left[\frac{m-u_{j+1}}{q_j}+1\right]^+\\&=S_j(bq_j)+lbp_j+S_j(l)+(-1)^j\left[\frac{m-u_{j+1}}{q_j}+1\right]^+\\&=S_j(bq_j)+lbp_j+S(l)+(-1)^j\left(\left[\frac{m-u_{j+1}}{q_j}+1\right]^+-\left[\frac{l-u_{j+1}}{q_j}+1\right]^+\right).\end{aligned}$$

In particular, if we write $m=z_1q_0+z_2q_1+\cdots+z_tq_{t-1}$ (with m_i denoting the ith subsum), and repeatedly apply the above with $b=z_{j+1}$ and $l=m_j$ for $j=t-1$ to 0, we obtain

$$S(m)=\sum_{i=1}^{t}\left(S_{i-1}(z_iq_{i-1})+m_{i-1}z_ip_{i-1}+(-1)^{i-1}I_i(m)\right),$$

where

$$\begin{aligned}I_i(m_i)=I_i(m_i,\alpha,\gamma)&:=\left[\frac{m_i-u_i}{q_{i-1}}+1\right]^+-\left[\frac{m_{i-1}-u_i}{q_{i-1}}+1\right]^+\\&=\left(z_i-\left[\frac{u_i-m_{i-1}}{q_{i-1}}\right]_*^+\right)^+.\end{aligned}$$

Now the $S_j(bq_j)$ are not difficult to evaluate. Indeed,

$$\begin{aligned} S_j(bq_j) &= \frac{1}{2}\, bq_j(bq_j+1)\,\frac{p_j}{q_j} + bq_j\gamma - \sum_{1\leqslant n\leqslant bq_j} \left\{ n\,\frac{p_j}{q_j}+\gamma \right\} \\ &= \frac{1}{2}\, b(bq_j+1)\,p_j + bq_j\gamma - b \sum_{1\leqslant n\leqslant q_j} \left\{ n\,\frac{p_j}{q_j}+\gamma \right\} \end{aligned}$$

where, with d_i and δ_i as defined in (7),

$$\begin{aligned} \sum_{1\leqslant n\leqslant q_j} \left\{ n\,\frac{p_j}{q_j}+\gamma \right\} &= \sum_{1\leqslant n\leqslant q_j} \left\{ \frac{np_j+d_{j+1}+\delta_{j+1}}{q_j} \right\} \\ &= \sum_{0\leqslant a\leqslant q_j-1} \left\{ \frac{a+\delta_{j+1}}{q_j} \right\} = \sum_{0\leqslant a\leqslant q_j-1} \frac{a+\delta_{j+1}}{q_j} \\ &= \frac{1}{2}\,(q_j-1)+\delta_{j+1}. \end{aligned}$$

So

$$S_{i-1}(z_i q_{i-1}) = \tfrac{1}{2} z_i\, p_{i-1}(z_i q_{i-1}+1) + z_i(\tfrac{1}{2}-\delta_i) + z_i q_{i-1}(\gamma-\tfrac{1}{2})$$

and

$$\begin{aligned} S(m) = {} & \sum_{i=1}^{t} \tfrac{1}{2} z_i\, p_{i-1}(z_i q_{i-1}+1+2m_{i-1}) \\ & + \sum_{i=1}^{t} (z_i(\tfrac{1}{2}-\delta_i) - (-1)^i\, I_i(m)) + (\gamma-\tfrac{1}{2})\, m. \end{aligned}$$

Now the first sum may be rewritten in terms of α rather than p_{i-1}:

$$\begin{aligned} &= \sum_{i=1}^{t} \tfrac{1}{2} z_i(q_{i-1}\alpha-\varepsilon_{i-1})(z_i q_{i-1}+1+2m_{i-1}) \\ &= \tfrac{1}{2}\alpha \left(\sum_{i=1}^{t} (z_i q_{i-1})^2 + 2 \sum_{i=1}^{t} \sum_{j=1}^{i-1} z_i q_{i-1} z_j q_{j-1} \right) + \tfrac{1}{2} m\alpha \\ &\quad - \tfrac{1}{2} \sum_{i=1}^{t} z_i \varepsilon_{i-1}(z_i q_{i-1}+1+2m_{i-1}) \\ &= \tfrac{1}{2} m(m+1)\alpha - \tfrac{1}{2} \sum_{i=1}^{t} z_i \varepsilon_{i-1}(z_i q_{i-1}+1+2m_{i-1}). \end{aligned}$$

Hence, finally

$$\begin{aligned} C_m(\alpha, \gamma) &= \sum_{1 \leqslant n \leqslant m} ((n\alpha+\gamma) - [n\alpha+\gamma] - \tfrac{1}{2}) \\ &= \tfrac{1}{2} m(m+1)\,\alpha + \gamma m - S(m) - \tfrac{1}{2} m \\ &= \sum_{i=1}^{t} (\tfrac{1}{2} z_i \varepsilon_{i-1}(z_i q_{i-1} + 1 + 2m_{i-1}) - z_i(\tfrac{1}{2} - \delta_i) + (-1)^i I_i(m)). \quad \blacksquare \end{aligned}$$

Proof of Corollary 1. Immediate from the trivial bounds

$$0 \leqslant \left(z_i - \left[\frac{u_i - m_{i-1}}{q_{i-1}} \right]_*^+ \right)^+ \leqslant z_i, \qquad -\frac{1}{2} z_i \leqslant z_i \left(\beta_i - \frac{1}{2} \right) \leqslant \frac{1}{2} z_i$$

and, recalling (2), the rough estimation

$$0 \leqslant \tfrac{1}{2} z_i \, |\varepsilon_{i-1}| \, (m_i + m_{i-1} + 1) < \tfrac{1}{2} z_i. \quad \blacksquare$$

Proof of Corollary 2. We first use the expansions of Proposition 3 to approximate M_n by the more predictable function

$$F_n = -\frac{1}{2a_n} z_n^2 + \left(\frac{b_n}{a_n} - \frac{1}{2} \right) z_n + (z_n - b_n)^+$$

where

$$b_n = \begin{cases} c_n & \text{if } n \text{ is even,} \\ a_n - c_n & \text{if } n \text{ is odd.} \end{cases}$$

Since (with the notations of Proposition 3) for $0 \leqslant z_n \leqslant a_n$

$$F_n \equiv -\frac{1}{2a_n} z_n^2 + \left(l_n + (-1)^n \frac{c_n}{a_n} - \frac{1}{2} \right) z_n + (z_n - (l_n a_n + (-1)^n c_n))^+,$$

we can write

$$M_n = F_n + A_1 + A_2 + A_3$$

where

$$A_1 = \left(\frac{1}{2} z_n - (-1)^n c_n \right) (q_{n-2} + \alpha_n q_{n-1}) \frac{z_n}{a_n} |\varepsilon_{n-1}|,$$

$$A_2 = \left((-1)^n (\gamma_n q_{n-1} - v_{n-1}) - m_{n-1} - \frac{1}{2} \right) z_n \, |\varepsilon_{n-1}|,$$

$$A_3 = (z_n - (l_n a_n + (-1)^n c_n + U)^+)^+ - (z_n - (l_n a_n + (-1)^n c_n))^+,$$

with

$$U=\left[\frac{l_n q_{n-2}+(\lambda_n+l_{n+1})\, q_{n-1}-(-1)^n v_{n-1}-m_{n-1}}{q_{n-1}}\right]_*.$$

We shall show that

$$-\left(3-\frac{1}{2a_n}\right)\leqslant M_n-F_n\leqslant\left(2-\frac{1}{2a_n}\right).$$

The proof is rather tedious and could be shortened at the cost of less precise constants.

We note the elementary estimates

$$\frac{1}{2}\, a_n(q_{n-2}+\alpha_n q_{n-1})\, |\varepsilon_{n-1}|\leqslant\frac{a_n}{a_n+2}\leqslant 1-\frac{2}{3}\, a_n^{-1},$$

$$a_n q_{n-2}\, |\varepsilon_{n-1}|\leqslant\frac{a_n}{a_n+1}\leqslant 1-\frac{1}{2}\, a_n^{-1}.$$

We first suppose that n is even.

If $l_n=1$ then $c_n=0$, $\gamma q_{n-1}-v_{n-1}<0$, $l_{n+1}+\lambda_n=1$ or 2 and the bounds follow from the rough estimates $0\leqslant A_1\leqslant(1-\frac{2}{3}a_n^{-1})$, $-2<A_2<0$, $A_3=0$ or 1.

If $l_n=0$ then $l_{n+1}+\lambda_n=1$ and, writing $u:=(q_{n-1}-v_{n-1}-m_{n-1}-\frac{1}{2})/q_{n-1}$, the lower bound follows easily from the inequalities $-1<u<2$, $A_2\geqslant-(1-u)^+$, $A_3\geqslant-[u]^+$ and, since $(\frac{1}{2}z_n-c_n)\, z_n\geqslant-\frac{1}{2}a_n^2$, $A_1\geqslant-(1-\frac{2}{3}a_n^{-1})$.

Now for $z_n\leqslant c_n$ we have $A_1\leqslant 0$, $A_2\leqslant u\, |\varepsilon_{n-1}|\, z_n q_{n-1}$ and $A_3\leqslant-[u]^-$, with $-1<u<1+q_{n-2}/q_{n-1}$ giving $A_1+A_2+A_3<1+a_n q_{n-2}\, |\varepsilon_{n-1}|\leqslant(2-\frac{1}{2}a_n^{-1})$.

For $z_n>c_n$ we have $A_1<(1-\frac{2}{3}a_n^{-1})$, $A_2\leqslant 1+[u]$, $A_3\leqslant-[u]$ and the upper bound is plain.

Next, suppose that n is odd.

If $l_n=0$ then $c_n=0$, $\gamma_n q_{n-1}-v_{n-1}<0$ and $\lambda_n+l_{n+1}=0$. Hence $0\leqslant A_1<(1-\frac{2}{3}a_n^{-1})$, $-1<A_2<1$, $A_3=0$, and the bounds are clear.

When $l_n=1$, we have $A_1\geqslant 0$. Set $w:=[(q_{n-2}+q_{n-1}+v_{n-1}-m_{n-1}-\frac{1}{2})/q_{n-1}]$. Then $-1\leqslant w\leqslant 2$, and the inequalities $A_2\geqslant-(2-w)-z_n q_{n-2}\, |\varepsilon_{n-1}|$, $A_3\geqslant-w$, lead to the lower bound $A_1+A_2+A_3\geqslant-(3-\frac{1}{2}a_n^{-1})$.

When $z_n\leqslant(a_n-c_n)$ we have $A_1\leqslant(1-\frac{2}{3}a_n^{-1})$. The inequalities $A_2\leqslant((v_{n-1}-m_{n-1})/q_{n-1})^+$ and $A_3\leqslant-[(q_{n-2}+v_{n-1}-m_{n-1})/q_{n-1}]_*^-$ readily yield $A_2+A_3<1$ and thus the upper bound in this case.

Clearly, $-1 \leqslant U \leqslant 2$. Hence when $z_n \geqslant (a_n - c_n + 2)$, or $z_n = (a_n - c_n + 1)$ and $U \neq 2$, we have $A_3 \leqslant -U$. Since $\lambda_n + l_{n+1}$ is 1 or 0 as $\gamma_n < \alpha_n$ or $\gamma_n \geqslant \alpha_n$ we also have $A_2 \leqslant 1 + U - (q_{n-2} + \alpha_n q_{n-1})\, z_n\, |\varepsilon_{n-1}|$. Plainly, $A_1 \leqslant \frac{3}{2}(q_{n-2} + \alpha_n q_{n-1})\, z_n\, |\varepsilon_{n-1}|$, and so $A_1 + A_2 + A_3 \leqslant 1 + (1 - \frac{2}{3} a_n^{-1})$ in these cases too.

Finally, when $z_n = (a_n - c_n + 1)$ and $U = 2$ we have $A_3 = -1$, $A_2 \leqslant 1$, and the upper bound follows on observing that, since $c_n \geqslant 1$, $A_1 \leqslant \frac{1}{2}(a_n + 2)$ $(q_{n-2} + \alpha_n q_{n-1})\, |\varepsilon_{n-1}| \leqslant 1$.

The first expressions in Corollary 2 then follow since (varying z_n, $0 \leqslant z_n \leqslant a_n$) F_n has minimum value

$$-\frac{1}{2}\frac{b_n}{a_n}\left(1 - \frac{b_n}{a_n}\right) a_n,$$

achieved at $z_n = b_n$ (or equivalently at $z_n = 0$ if $b_n = a_n$) and maximum

$$\frac{1}{2}\left(\frac{b_n}{a_n} - \frac{1}{2}\right)^2 a_n - \frac{1}{2a_n}\left\{\frac{a_n}{2}\right\}^2$$

achieved at $z_n = [b_n \pm \frac{1}{2} a_n]$. Since we can take these extremal values with $z_n < a_n$ we can find m that maximise or minimise the $(-1)^n F_n$ simultaneously.

The second expresssions follow on observing that for varying b_n the function F_n has maximum and minimum value

$$\frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i, \qquad -\frac{z_i}{a_i}\left(1 - \frac{z_i}{a_i}\right) a_i$$

achieved at $b_n = a_n$ or 0 and at $b_n = z_n$ (equivalently at $b_n = a_n$ or 0 if $z_n = a_n$ or 0) respectively.

The bounds for the discrepancy in Corollary 5 arise by similarly showing that $-3 < H_n - f_n < 2$, where

$$f_n = -\left(1 - \frac{b_n}{a_n}\right) a_n + (z_n - b_n)^+,$$

with some gain from the trivial bound when $z_n = 0$ (the maximum of f_n). ∎

Proof of Corollary 3. Clearly for $0 \leqslant c_i \leqslant a_i$ we have

$$\left(\frac{c_i}{a_i} - \frac{1}{2}\right)^2 \leqslant \frac{1}{4}, \qquad \frac{c_i}{a_i}\left(1 - \frac{c_i}{a_i}\right) \leqslant \frac{1}{4}.$$

Hence from Corollary 2

$$\max_{1 \leqslant m < q_t} |C_m(\alpha, \gamma)| \leqslant \tfrac{1}{8} \sum_{i=1}^{t} (a_i + 20) + \tfrac{1}{2}.$$

From the proof of Corollary 2 we also have

$$\begin{aligned}
&2 \max_{m < q_t} |C_m(\alpha, \gamma)| \\
&\quad \geqslant \max_{m < q_t} C_m(\alpha, \gamma) - \min_{m < q_t} C_m(\alpha, \gamma) \\
&\quad \geqslant \sum_{n=1}^{t} \left\{ \left(\frac{1}{2} \left(\frac{b_n}{a_n} - \frac{1}{2} \right)^2 a_n - 3 \right)^+ - \left(-\frac{1}{2} \frac{b_n}{a_n} \left(1 - \frac{b_n}{a_n} \right) a_n + 2 \right)^- \right\} \\
&\quad \geqslant \sum_{n=1}^{t} \left(\frac{1}{8} a_n - 5 \right)^+ . \quad \blacksquare
\end{aligned}$$

Proof of Property 1(a). Suppose that $m = z_1 q_0 + \cdots + z_t q_{t-1}$, $z_t \neq 0$. Then, since the slowest growth in denominators occurs for the Fibonacci numbers F_n,

$$m \geqslant q_{t-1} \geqslant F_t \geqslant \left(\frac{1+\sqrt{5}}{2} \right)^{t-2} \Rightarrow t \leqslant \frac{\log\left(\dfrac{3+\sqrt{5}}{2} m \right)}{\log\left(\dfrac{1+\sqrt{5}}{2} \right)}.$$

Hence if $\sum_{i=1}^{t} a_i \leqslant At$ then, from Corollary 3,

$$|C_m(\alpha, \gamma)| \leqslant \frac{1}{8} (A+24) t \leqslant \frac{(A+24)}{8 \log\left(\dfrac{1+\sqrt{5}}{2} \right)} \log\left(\frac{3+\sqrt{5}}{2} m \right). \quad \blacksquare$$

Proof of Property 1(b). By a lemma of Ostrowski [14, pp. 85–86] it is certainly true that $\sum_{i=1}^{t} a_i > \log q_t$. Hence, if $\sum_{i=1}^{t} a_i \geqslant (B+40) t$, Corollary 3 gives

$$\begin{aligned}
\max_{m < q_t} |C_m(\alpha, \gamma)| &\geqslant \frac{1}{16} \sum_{i=1}^{t} (a_i - 40) \geqslant \frac{1}{16} \max(Bt, \log q_t - 40t) \\
&\geqslant \frac{1}{16} \frac{B}{B+40} \log q_t .
\end{aligned}$$

The bounds stated when $a_i \leqslant A$ or $a_i \geqslant B$ arise similarly on noting that

$$(a_i+25) \leqslant \frac{A}{\log A} \log a_i + 26, \qquad (a_i - 9) \geqslant \frac{(B-9)}{\log(B+1)} \log(a_i+1)$$

together with the trivial bounds $\prod_{i \leqslant t} a_i \leqslant q_t \leqslant \prod_{i \leqslant t} (a_i+1)$. ∎

Proof of Property 2(a). For any convergent denominator q_s Corollary 1 gives

$$|C_m(\alpha, \gamma)| \leqslant \frac{3}{2} \sum_{i=1}^{t} z_i \leqslant \frac{3}{2} \left(\sum_{i=1}^{s} z_i q_{i-1} + \frac{1}{q_s} \sum_{i=s+1}^{t} z_i q_{i-1} \right) \leqslant \frac{3}{2} \left(q_s + \frac{m}{q_s} \right)$$

and the result follows on picking q_s to be the largest convergent less than $\sqrt{m}$. Plainly for fixed, irrational α, such a bound is $o(m)$ (since $\max\{q_s < \sqrt{m}\} \to \infty$ as $m \to \infty$). ∎

Proof of Property 2(b). From Corollary 3 we have

$$\max_{m \leqslant q_t} |C(m)| \geqslant \frac{1}{16} (a_t - 40) \geqslant \frac{1}{16} \left(\frac{q_t}{q_{t-1}} - 41 \right) \tag{16}$$

and this is not $o(m)$ uniformly in α. Specifically, given any $f(n) = o(n)$ we can generate α with the desired behaviour by (iteratively) choosing the partial quotients to satisfy

$$a_{j+1} \geqslant \min \left\{ n \geqslant 82 q_j : \left| \frac{f(m)}{m} \right| \leqslant \frac{1}{32 q_j \, g(j)} \text{ for all } m \geqslant \frac{n}{16} \right\},$$

where $g(j)$ can be any function such that $g(j) \to \infty$ as $j \to \infty$. From (16) we know that there exists an $m < q_{j+1}$ with

$$|C_m(\alpha, \gamma)| \geqslant \frac{1}{16} \left(\frac{q_{j+1}}{q_j} - 41 \right) \geqslant \frac{1}{32} \frac{q_{j+1}}{q_j},$$

where the trivial bounds $\frac{1}{2} m \geqslant |C_m(\alpha, \gamma)| \geqslant \frac{1}{32} a_{j+1}$ ensure that m is sufficiently large that (by the definition of a_{j+1})

$$\max_{m < q_{j+1}} \left| \frac{C_m(\alpha, \gamma)}{f(m)} \right| \geqslant \frac{1}{32} \frac{q_{j+1}}{q_j} \frac{32 q_j \, g(j)}{m} > g(j) \to \infty, \qquad \text{as} \quad j \to \infty. \ ∎$$

Proof of Property 3(a). Observe that

$$\frac{1}{f(q_{i-1}) \, q_{i-1}^{1+r}} < \| q_{i-1} \alpha \| < \frac{1}{q_i} \Rightarrow q_{i-1} > \left(\frac{q_i}{f(q_{i-1})} \right)^{1/1+r}.$$

Hence if $z_i \neq 0$ then $q_{i-1} \leqslant m_i < q_i$ and

$$z_i \leqslant \frac{m_i}{q_{i-1}} < m_i \left(\frac{f(q_{i-1})}{q_i} \right)^{1/1+r} < m_i^{r/1+r} f(m_i)^{1/1+r}.$$

Thus, by Corollary 1,

$$|C_m(\alpha, \gamma)| \leqslant \frac{3}{2} \sum_{i=1}^{t} z_i \leqslant \frac{3}{2} m^{1/1+r} f(m)^{1/1+r} t$$

$$\leqslant \frac{3 \log \left(\frac{3+\sqrt{5}}{2} m \right)}{2 \log \left(\frac{1+\sqrt{5}}{2} \right)} m^{1/1+r} f(m)^{1/1+r}. \quad \blacksquare$$

Proof of Property 3(b). Suppose that for a suitably large q (large enough that $q^{r/1+r} \geqslant 164$) we have $q^{1+r} \|q\alpha\| \leqslant 1$. Then q must be a convergent (q_{t-1} say) with

$$\frac{1}{2q_t} < \|q_{t-1}\alpha\| < \frac{1}{q_{t-1}^{1+r}} \Rightarrow q_{i-1} < 2q_t^{1/1+r}.$$

Hence by Corollary 4(b) there exists an $m < q_t$ with

$$|C_m(\alpha, \gamma)| \geqslant \frac{1}{16} \left(\frac{q_t}{q_{t-1}} - 41 \right) \geqslant \frac{1}{64} q_t^{r/1+r} \geqslant \frac{1}{64} m^{r/1+r}.$$

Since the bound grows with q_t we clearly generate infinitely many distinct m in this way. $\blacksquare$

Proof of Corollary 6. Suppose that $\sum_{i=1}^{t} a_i \leqslant At$ for some $t \geqslant (3168A + 1152)$. Then there are certainly at least $[t/4]$ partial quotients $a_{2j} \leqslant 4A$, $2j \leqslant t$. From these we can select a subsequence of at least $N = [t/8]$ with $a_{2n_i} \leqslant 4A$ and $n_{i+1} - n_i \geqslant 2$ for each $1 \leqslant i \leqslant N$.

Taking $m = q_{2n_1-2} + q_{2n_2-2} + \cdots + q_{2n_N-2} < q_t$ we observe that

$$m_{2n_i-1} + m_{2n_i-2} + 1 = q_{2n_i-2} + 2m_{2n_i-5} + 1 \leqslant q_{2n_i-2} + q_{2n_i-3}$$

and

$$|\varepsilon_{2n_i-2}|^{-1} = q_{2n_i-1} + \alpha_{2n_i-1} q_{2n_i-2} \geqslant (q_{2n_i-2} + q_{2n_i-3})(1 + \tfrac{1}{2}\alpha_{2n_i-1})$$

where

$$\alpha_{2n_i-1}=\frac{1}{a_{2n_i}+\alpha_{2n_i}}>\frac{1}{4A+1}.$$

Hence, since plainly $A\geqslant 1$, (9), (11), and (14) give

$$\begin{aligned}\Delta_m(1-\gamma',\alpha)&=-\{\gamma'\}+\sum_{i=1}^{[t/8]}\left(1-|\varepsilon_{2n_i-2}|\,(m_{2n_i-1}+m_{2n_i-2}+1)\right)\\&\geqslant -1+\left[\frac{t}{8}\right]\frac{5}{11(4A+1)}\geqslant\frac{1}{18A(4A+1)}\sum_{i=1}^{t}a_i\\&>\frac{1}{18A(4A+1)}\log q_t.\end{aligned}$$

The lower bound follows on reversing the roles of odd and even. ∎

Proof of Proposition 1. With d_i and δ_i as in (7), writing

$$\left[k\frac{p_{n-1}}{q_{n-1}}+\gamma\right]=\left[\frac{kp_{n-1}+d_n+\delta_n}{q_{n-1}}\right]=\left[\frac{kp_{n-1}+d_n}{q_{n-1}}\right]$$

and

$$[k\alpha+\gamma]=\begin{cases}\left[\dfrac{kp_{n-1}+d_n+(\delta_n+k\,|\varepsilon_{n-1}|)}{q_{n-1}}\right] & \text{if } n \text{ is odd,}\\[2ex] \left[\dfrac{kp_{n-1}+d_n+(\delta_n-k\,|\varepsilon_{n-1}|)}{q_{n-1}}\right] & \text{if } n \text{ is even.}\end{cases}$$

it is not hard to see that,

$$u_n=\min\{k\in\mathbb{N}\colon \delta_n+k\,|\varepsilon_{n-1}|\geqslant 1 \text{ and } kp_{n-1}+d_n+1\equiv 0 \pmod{q_{n-1}}\}$$

if n is odd, and

$$u_n=\min\{k\in\mathbb{N}:\delta_n-k\,|\varepsilon_{n-1}|<0\ \text{ and } kp_{n-1}+d_n\equiv 0\ \pmod{q_{n-1}}\}.$$

if n is even.

Hence, recalling the familiar identity $q_n p_{n-1}-q_{n-1}p_n=(-1)^n$, we readily see that, when n is odd

$$\begin{aligned} u_n &= \min\{q_n(d_n+1) - sq_{n-1} : (q_n(d_n+1) - sq_{n-1}) \geqslant (1-\delta_n)\,|\varepsilon_{n-1}|^{-1}\} \\ &= q_n(d_n+1) - q_{n-1}[\gamma q_n - \alpha_n \beta_n] \\ &= q_n(d_n+1) - q_{n-1}(d_{n+1} - \lambda_n) \\ &= \beta_n q_n + (\beta_{n+1} + \lambda_n)\, q_{n-1}. \end{aligned}$$

Similarly, when n is even

$$\begin{aligned} u_n &= \min\{tq_{n-1} - d_n q_n : (tq_{n-1} - d_n q_n) > \delta_n\,|\varepsilon_{n-1}|^{-1}\} \\ &= ([\gamma q_n + \alpha_n \beta_n] + 1)\, q_{n-1} - d_n q_n \\ &= (d_{n+1} + \lambda_n + 1)\, q_{n-1} - d_n q_n \\ &= \beta_n q_n + (\beta_{n+1} + \lambda_n)\, q_{n-1}. \quad \blacksquare \end{aligned}$$

Proof of Proposition 2. Expression (6) is a straightforward exercise in induction. Property (i) is immediate from the definition of the c_i, and property (ii) amounts to the inequality

$$\frac{\gamma_n}{\alpha_n} = \frac{\gamma_{n-1} - \alpha_{n-1} c_n}{1 - \alpha_{n-1} a_n} < 1$$

if $a_n = c_n$.

Property (iii) holds since $c_{k+2i} = a_{k+2i}$, $c_{k+2i+1} = 0$ for all $i \geqslant 0$ would (by (6)) imply that

$$|\varepsilon_{k-2}| > \gamma_{k-1}\,|\varepsilon_{k-2}| = \sum_{i=k}^{\infty} c_i\,|\varepsilon_{i-1}| = |\varepsilon_{k-2}|.$$

To see the uniqueness, suppose that we have two representations

$$\sum_{i=1}^{\infty} b_i\,|\varepsilon_{i-1}| = \gamma = \sum_{i=1}^{\infty} b'_i\,|\varepsilon_{i-1}|$$

where the b_i and b'_i both satisfy (i), (ii) and (iii) with $b_k > b'_k$ and $b_j = b'_j$, $j < k$. By (iii) there must exist an $I \geqslant 0$ such that $b'_{k+2I+1} \neq a_{k+2I+1}$ and $b_{k+2i-1} = a_{k+2i-1}$, $b_{k+2i} = 0$ for all $1 \leqslant i \leqslant I$ and (since $b'_j < a_j$ for infinitely many succeeding j) we obtain the false inequality:

$$\begin{aligned} |\varepsilon_{k-1}| &\leqslant \sum_{i>k} (b'_i - b_i)\,|\varepsilon_{i-1}| \\ &< \sum_{i=1}^{I} a_{k+2i-1}\,|\varepsilon_{k+2i-2}| + \sum_{j=1}^{\infty} a_{k+2I+j}\,|\varepsilon_{k+2I+j-1}| - |\varepsilon_{k+2I}| \\ &= |\varepsilon_{k-1}|. \quad \blacksquare \end{aligned}$$

Proof of Proposition 3. The first relation follows from the observation that

$$\{\gamma q_{n-1}\} = \{\{(-1)^{n-1} v_n \alpha + \gamma_n \,|\varepsilon_{n-1}|\}\, q_{n-1}\}$$
$$= \{(v_n + \gamma_n q_{n-1})\,|\varepsilon_{n-1}|\}.$$

where $-1 < (v_n + \gamma_n q_{n-1})\,|\varepsilon_{n-1}| < 1$ since

$$-q_{n-1} \leqslant (v_n + \gamma_n q_{n-1}) \leqslant q_n - (a_n - c_n - \gamma_n)\, q_{n-1} < q_n + \alpha_n q_{n-1}.$$

The expression for u_n then comes from merely substituting this in Proposition 1 and observing that $q_n(\gamma_n/\alpha_n - \gamma_{n+1}) = c_{n+1} q_n = (v_{n+1} + v_n)$. ▮

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