About Hopf-type theorems for f-neighbors

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Overview

f-neighbors

Definition

Let $f : M \to \mathbb{R}^n$ be a continuous mapping of a metric space (M, d) with metric d. Distinct points a and b in M are called:

- *neighbors*: iff $f(a) = f(b)$
- *spherical neighbors*: iff $f(a) = f(b)$ or $\exists$ a euclidean ball $B^n \subset \mathbb{R}^n : \partial B^n \supset f(a), f(b)$ and $\mathrm{int} B^n \cap f(M) = \emptyset$
- *visual neighbors*: iff $f(a) = f(b)$ or the segment with endpoints $f(a)$ and $f(b)$ intersect $f(M)$ only at these two points
- *topological neighbors*: iff $f(a) = f(b)$ or some path in $\mathbb{R}^n$ with endpoints at $f(a)$ and $f(b)$ intersect $f(M)$ only at these two points

Hopf Theorem

Definition

Let:

1. $\Omega_f = \{r \in \mathbb{R} \mid \exists$ neighbors a and b in M with $d(a,b) = r\}$.
2. $\Omega_f^{sph} = \{r \in \mathbb{R} \mid \exists$ spherical neighbors a and b in M with $d(a,b) = r\}$.
3. $\Omega_f^{vis} = \{r \in \mathbb{R} \mid \exists$ visual neighbors a and b in M with $d(a,b) = r\}$.
4. $\Omega_f^{top} = \{r \in \mathbb{R} \mid \exists$ topological neighbors a and b in M with $d(a,b) = r\}$.

Then $\Omega_f \subset \Omega_f^{sph} \subset \Omega_f^{vis} \subset \Omega_f^{top}$

Hopf type theorems

Theorem (K. Borsuk, S. Ulam, 1933)

Under any continuous map $f : \mathbb{S}^n \to \mathbb{R}^n$ some two opposite points are mapped to a single point.

Theorem (H. Hopf, 1944)

Let M be a compact Riemannian manifold of dimension n and $f : M \to \mathbb{R}^n$ be a continuous map. Then for any prescribed $\varepsilon > 0$ there exists a pair $x, y \in M$ such that $f(x) = f(y)$ and the points x and y are connected by a geodesic of length δ.

Theorem (O.R.Musin, A.V.Malutin, 2021)

Let $\mathbb{S}^n$ be a unit sphere, and let $\mathbb{S}^n \to \mathbb{R}^m$ be a continuous map. Then there are spherical neighbors x and y in $\mathbb{S}^n$ with $d(x,y) \geq \sqrt{2(n+2)/(n+1)}$.

Problem

Find and describe non-trivial properties of the sets $\Omega_f^{sph} \subset \Omega_f^{vis} \subset \Omega_f^{top}$ for the case of mapping $f : \mathbb{S}^n \to \mathbb{R}^m$ with $m > n$.

Theorem (Example 1)

Let $k = \{1, 2\}$. For any prescribed $\alpha \in (0, \pi]$ there exists a continuous map $f_\alpha : \mathbb{S}^k \to \mathbb{R}^{k+1}$ such that $\Omega^{vis}_{f_\alpha} \not\ni \alpha$.

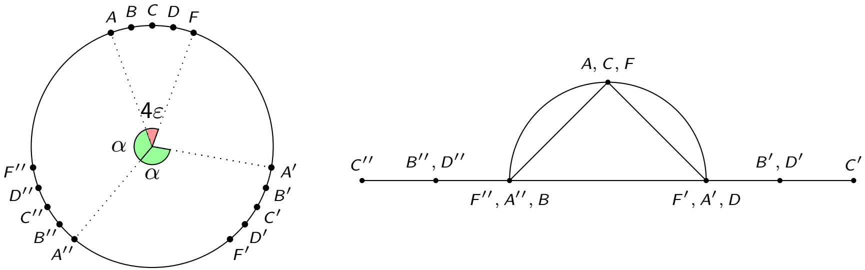

Figure 1. Example $f_\alpha : \mathbb{S}^1$ left and $f_\alpha(\mathbb{S}^1)$ right

Example 1. Proof by „coloring“

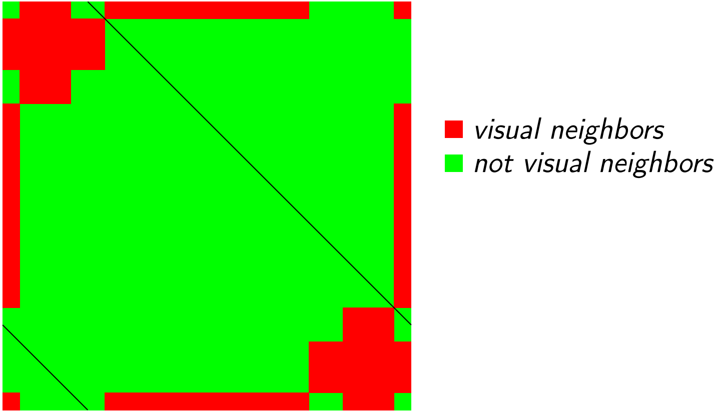

Figure 2. The set of pairs of points $S^1 \times S^1 = T^2$.

Let μ be the Lebesgue measure on $\mathbb{R}$.

Theorem (Example 2. Torus knot diagrams)

There exists a sequence of continuous maps $f_n : \mathbb{S}^1 \to \mathbb{R}^2$ such that $\lim_{n\to\infty} \mu(\Omega^{vis}_{f_n}) = \lim_{n\to\infty} \mu(\Omega^{top}_{f_n}) = 0.$

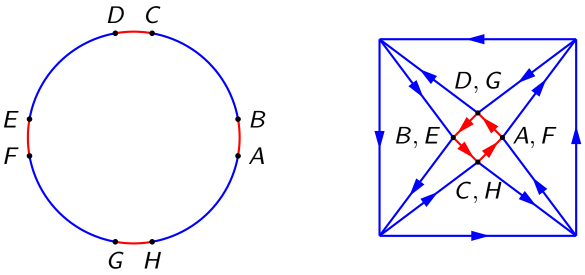

Figure 3. Example of f_4: $\mathbb{S}^1$ left and $f_\alpha(\mathbb{S}^1)$ right, which is a (3,4)-torus knot diagram.

Example 2. Proof by „coloring“

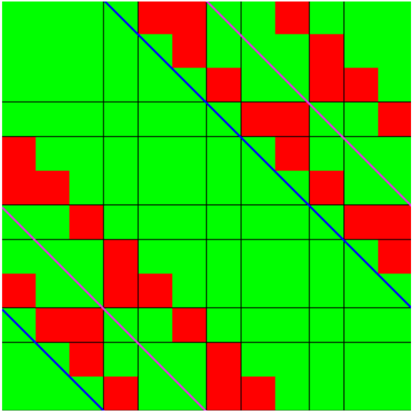

Figure 4. The set of pairs of points $S^1 \times S^1 = T^2$.

Theorem (Example 3. „Distending spheres")

There exists a sequence of continuous maps $f_n : \mathbb{S}^2 \to \mathbb{R}^3$ such that $\lim_{n\to\infty} \mu(\Omega^{vis}_{f_n}) = 0$.

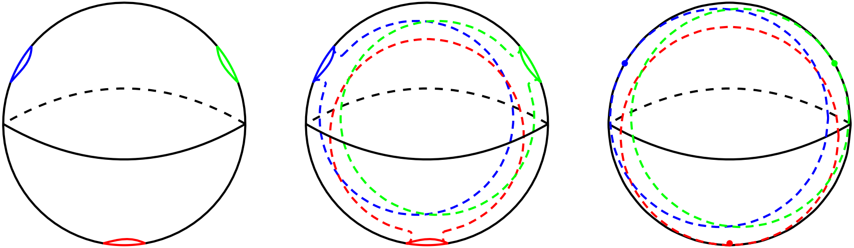

Figure 5. Example of f_3: $\mathbb{S}^2$ left and $f_\alpha(\mathbb{S}^2)$ right

Theorem (Example 4. „Trefoil“)

For any prescribed $\varepsilon > 0$ *there exists a continuous map* $f : \mathbb{S}^1 \to \mathbb{R}^2$ *such that* $\Omega_f^{vis} = [0, \varepsilon) \cup (\frac{2\pi}{3} - \varepsilon, \pi]$.

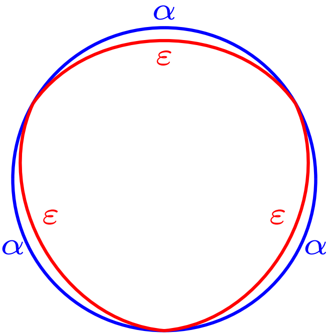

Figure 6. $f(\mathbb{S}^1)$

Theorem

Let $f : \mathbb{S}^n \to \mathbb{R}^m$ be a continuous map with $m > n$. Then Ω_f^{vis} is at least countable.

Proof.

If f is not a Peano curve, i.e. $\mathrm{Int}(f(\mathbb{S}^1))$ is not empty, then the statement of the theorem is obvious.
Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a Peano curve. Without loss of generality let's suppose that $\mathbb{S}^1 = I^2$ is a unit square in $\mathbb{R}^2$. If Ω_f^{vis} is a finite set of numbers, take $x = \min\{\Omega_f^{vis}\}$. $[0, 2\pi] = \bigcup_{k=1}^{n=\lceil 2\pi/x \rceil} I_k$, where I_k are segments in [0,2π] and $\mathrm{Int}(I_k) \cap \mathrm{Int}(I_j) = \emptyset$ for $k \neq j$.

□

Proof.

Let's suppose that for every $f_k := f|_{I_k} \nexists \varepsilon > 0$ and a point $(x_0, y_0) \in I^2$ such that $f|_{I_k}$ fills all of the square with centre (x_0, y_0) and length of side ε. Then we get that f does not fill all points of I^2. Thus the image of some f_k fills some square, consequently f_k has there multiple points with angle distance different from Ω_f^{vis}.

□

Theorem (Special Peano case 1)

Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a Peano curve. If there exists such an arc δ in $\mathbb{S}^1$ that $Int(f(\delta)) = \emptyset$, then Ω_f^{vis} is uncountable.

Theorem (Special Peano case 2)

Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be a Peano curve and $\exists\, [a,b] \subset [0,2\pi]$ and $[c,d] \subset [0,2\pi] : [a,b] \cap [c,d] = \emptyset$ and $int(f([a,b] \cap f([c,d])) \neq \emptyset$, but $\nexists\, [a_1, b_1] \subseteq [a,b], [c_1, d_1] \subseteq [c,d] : f|_{[a_1,b_1]} \equiv f|_{[c_1,d_1]}$. Then Ω_f^{vis} is uncountable.

Recent results

Theorem

Let $f : \mathbb{S}^1 \to \mathbb{R}^2$ be continuous map. Then Ω_f^{sph} is at least countable.

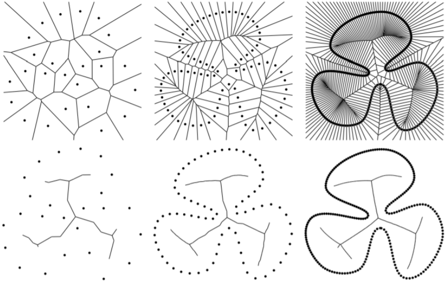

Figure 7. Notion of the Medial axis and Voronoy diagram

Recent results

Definition

Let $M = \{p \in M | \; \exists$ a set of distinct points $T_p = \{y_i\}_{i=1}^{N_p}$ on the ∂X with $N > 1$ such that $(p, y_i) = (p, \partial X), i = 1, ..., N_p\}$. Let $R_{p,i}, i = 1, ..., N_p$ be segment with begin in p and and in y_i. We call them radii emitted from a point p.

Definition

For each point $p \in M$ corresponds some set of points on the ∂X which lie on the same circle with the centre at p. Denote this circle as $\mathbb{S}^1_p$ and this radius as r_p. Let $\partial X_M = \bigcup_{p \in M} \mathbb{S}^1_p \cap \partial X$.

Recent results

Definition

Let $S = \{x_i\}$ be some spherical neighbors under map f. If this set is maximal on inclusion then we call the convex hull of the set $\{f(x_i)\}$ as Delaune tile.

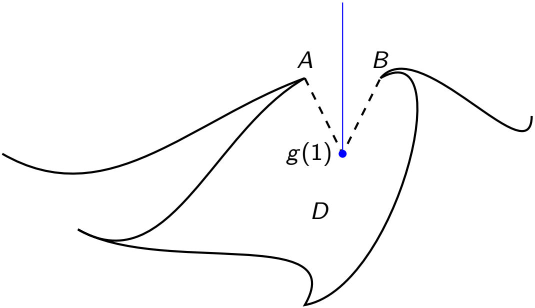

Figure 8. Part of the medial axis

Recent results

Let M^n be the Riemannian manifold of the dimension $n > 1$.

Definition

Let $\Phi(a) = \{A, B \in M : f(a) = f(b)$ and $\exists$ geodesic of the length a in M^n, connecting A and B $\}$

Theorem

Let $f : \mathbb{M}^n \to \mathbb{R}^n$ be the continuous map. Then if there is not conjugate points, which are connected by geodesic of length a, then $\Phi(a)$ is uncountable.

Definition

The Peano curve $f : \mathbb{S}^1 \to \mathbb{R}^2$ is called monotone iff $S(f(t))$ is strictly monotone function, where $t \in [0, 2\pi]$ is parametrization of $\mathbb{S}^1$.

Problem (1)

Is it true that for a monotone Peano curve Ω_f^{vis} is uncountable?

Problem (2)

Is it true that Ω_f^{sph} is infinite in general case?