CRITICAL HEREDITARY GRAPH CLASSES: A SURVEY OF RESULTS

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Hereditary graph classes

Definition

A class of graphs is any set of simple graphs (i.e. unlabelled, non-oriented graphs without loops and multiple edges), closed under isomorphism.

Definition

A class of graphs is hereditary if it is closed under deletion of vertices.

Properties and notation

Any hereditary class $\mathcal{X}$ can be defined by the set $\mathcal{Y}$ of its forbidden induced subgraphs (i.e. minimal graphs under vertex deletions, not belonging to $\mathcal{X}$); it is written as $\mathcal{X} = Free(\mathcal{Y})$.

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Examples

- $\mathcal{F}orests = Free(\{C_3, C_4, C_5, \ldots\})$
- $\mathcal{B}ipartite = Free(\{C_3, C_5, C_7, \ldots\})$
- $\mathcal{S}ums\ of\ Cliques = Free(\{P_3\})$

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A hereditary class $\mathcal{X}$ is finitely defined if the set of its forbidden induced subgraphs is finite.

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The classes $\mathcal{F}orests$ and $\mathcal{B}ipartite$ are not finitely defined.

The class $\mathcal{S}ums\ of\ Cliques$ is finitely defined. For any d, the class $\mathcal{D}eg(d)$ of graphs of the maximum degree at most d is finitely defined.

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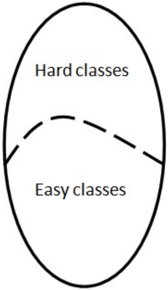
General problem statement

Problem

How to classify, for a given graph problem, hereditary classes into easy and hard instances under natural definitions of easiness and hardness?



Π-easy and Π-hard graph classes

Definition

Let Π be any NP-complete graph problem.

A hereditary class is said to be Π-easy if the problem Π can be solved in polynomial time for its graphs.

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A hereditary class, which is not Π-easy, is said to be Π-hard.

Assumption

We always assume that $\mathbf{P} \neq \mathbf{NP}$.

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Maximal easy and minimal hard classes

Result [straightforwardly]

For any Π, there are not maximal Π-easy classes.

Sketch of proof

From the contrary. If $\mathcal{X}$ is a Π-easy class, then $\mathcal{X}$ is distinct to the set of all graphs. Then there exists a graph $G \notin \mathcal{X}$. The class $\mathcal{X} \cup [\{G\}]_h$, where $[\cdot]_h$ is the hereditary closure of an argument, is also Π-easy, and $\mathcal{X}$ is not maximal.

Remark

Minimal hard classes exist for some graph problems and they do not exist for others.

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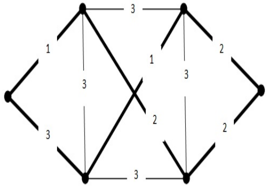
The Hamiltonian cycle is a cycle, once visiting all vertices of a graph. The travelling salesman problem is to find in a given edge-weighted graph a Hamiltonian cycle with the minimum sum of weights.



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The set of all complete graphs is a minimal hard class for the travelling salesman problem.

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The problem is NP-hard for the class of all the complete graphs. Any proper hereditary subclass of this class contains only a finite set of graphs, as being a set of complete graphs of a bounded size.

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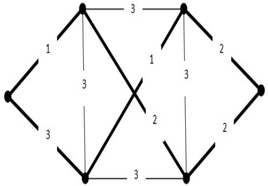

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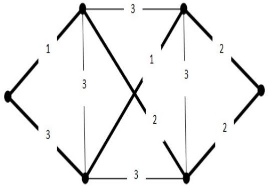

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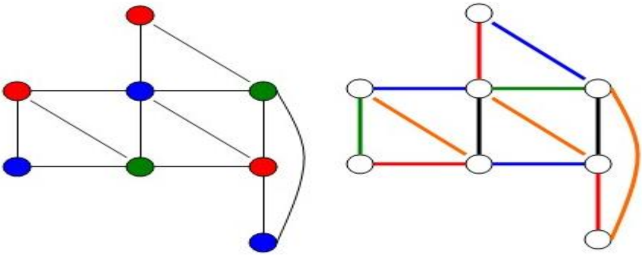
Definition

A proper vertex k-colouring of graph $G = (V, E)$ is a mapping $c: V \to \{1, 2, \ldots, k\}$, such that $c(v) \neq c(u)$, for any adjacent vertices v and u.

The vertex k-colourability problem is, for a given graph G, to check whether G has a proper vertex k-colouring or not. The edge k-colourability problem is defined in a similar way.



Result [M.'09]

For any k and the vertex and edge variants of the k-colourability problem, there are not minimal hard classes.

Sketch of proof

In any hereditary class $\mathcal{X}$, which is a hard case for the k-colourability problems, there is a graph $G \in \mathcal{X}$, which is not k-colourable. The set $\mathcal{X} \setminus Free(G)$ consists of graphs, each of which is not k-colourable. It is possible to determine in polynomial time whether a given graph from $\mathcal{X}$ belongs to $\mathcal{X} \setminus Free(G)$. Thus, there is a polynomial-time reducibility for the problems in $\mathcal{X}$ to the same problems for $\mathcal{X} \setminus Free(G)$.

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The absence of minimal hard classes is true for any NP-complete graphs recognition problem from a hereditary class, like the classes of k-colourable graphs, unit disk graphs, boxicity 2 graphs and etc.

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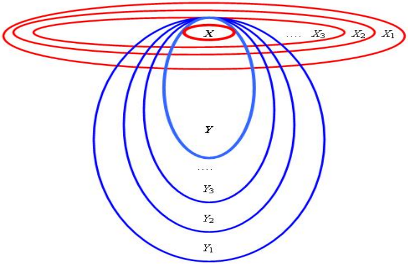
Definitions

A class of graphs $\mathcal{X}$ is called Π-limit, if there is an infinite sequence $\mathcal{X}_1 \supseteq \mathcal{X}_2 \supseteq \mathcal{X}_3 \supseteq \ldots$ of Π-hard classes, such that $\mathcal{X} = \bigcap_{i=1}^{\infty} \mathcal{X}_i$.

Any minimal Π-limit class is said to be Π-boundary.



Result [Alekseev'04]

A finitely defined class is Π-hard if and only if it contains some Π-boundary class.

Remark 1

Knowledge of all Π-boundary classes allows to classify the complexity of Π for finitely defined classes.

Remark 2

Maximal easy and minimal hard classes are boundary points, but boundary classes is a tool for classification of finitely defined classes.

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Boundary classes exist for any NP-complete graph problem.

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Examples of boundary classes for some graph problems

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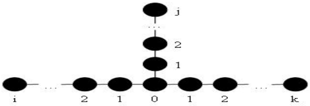
A triode $T_{i,j,k}$ $(i, j, k \geq 0)$ is a tree of the form, depicted on the picture.



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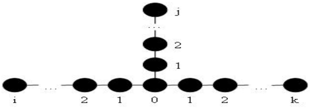

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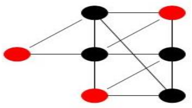

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The theorem of V.E. Alekseev is not true for hereditary graph classes, which are not finitely defined.

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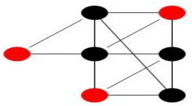

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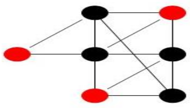

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Remark

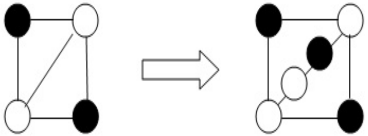
Any proof that a class is boundary can be split into two parts:

- A proof that the class is limit by presenting a sequence of hard classes, converging to it.
- A proof that the class is boundary by showing that in any monotonically descreasing chain of hereditary classes, converging to any its proper subclass, there is an easy element.

Sketch of proof(I): Fact №1

The double subdividing of any edge of any graph increases its independence number by one.

If H is the result of the $2k$-subdividing for each of m edges of a graph G, then $\alpha(H) = \alpha(G) + km$.



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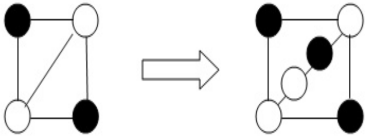

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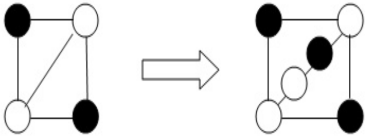

Sketch of proof(I): Fact №2

- The independent set problem is known to be NP-complete for subcubic graphs.
- For any k, the independent set problem is NP-complete for the class $\mathcal{X}_k$ of all the subcubic graphs, in which any two degree 3 vertices lye at the distance at least k.

Sketch of proof(I): Fact №3

The following inclusions and relation hold:

$$\mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \mathcal{Y}_3 \supset \ldots, \text{ where } \mathcal{Y}_k = \bigcup_{i=k}^{\infty} \mathcal{X}_k,$$

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Sketch of proof(II)

Let $\mathcal{Z}_1 \supseteq \mathcal{Z}_2 \supseteq \dots$ be any sequence of hard for the independent set problem hereditary classes, such that $\bigcap_{i=1}^{\infty} \mathcal{Z}_i \subset \mathcal{T}$. Then, there are a member $\mathcal{Z}_{i^*}$ and a graph $G \in \mathcal{T}$, such that $\mathcal{Z}_{i^*} \subseteq Free(\{G\})$.

Definition and property

A class of graphs is strongly hereditary (or monotone) if it is closed under deletion of vertices and edges.

Any monotone class can be defined by the set (finite or infinite) of its forbidden subgraphs.

Sketch of proof(II)

The monotone class $[\mathcal{Z}_{i^*}]_m$, where $[\cdot]_m$ is the monotone closure of an argument, does not contain $\mathcal{T}$.

Any monotone class, not containing $\mathcal{T}$, has a (uniformly) bounded clique-width (Lozin and Boliac; ISAAC 2002: 44-54). The independent set problem is polynomial-time solvable in any graph class with bounded clique-width (Courcelle, Makowsky, Rotics; Theory of Computing Systems 2000(33): 125–150).

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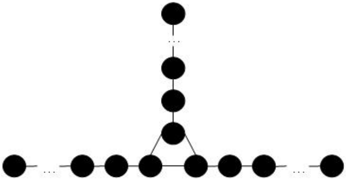

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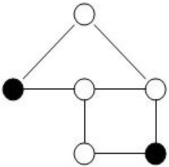
The dominating set problem is to find in a given graph $G = (V, E)$ a minimum subset $D \subseteq V$, such that any element of $V \setminus D$ has a neighbour in D.



Result [Alekseev, Korobitsyn, Lozin'04]

The classes $\mathcal{T}$ and $\mathcal{D}$ are boundary for the dominating set problem.

Remark

The classes $\mathcal{T}$ and $\mathcal{D}$ are simultaneously boundary for many graph problems (such as, the dominating set problem, the dissociating set problem, the maximum subgraph problem and etc.).

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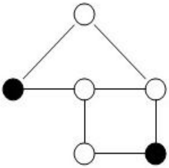

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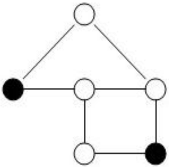

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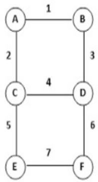
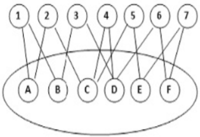
For a given graph $G = (V, E)$, a graph $Q(G)$ has:

- the set of vertices $V \cup E$
- the set of edges $\{(v_i, v_j) : \ v_i, v_j \in V\} \cup \{(v, e) : \ v \in V, e \in E, v$ is incident to $e\}$.





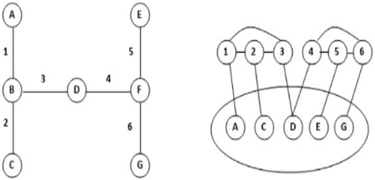
Definition

Let $G = (V, E)$ be a graph of maximum degree 3 and V' be the set of its degree 3 vertices. A graph $Q^*(G)$ has:

- the set of vertices $(V \setminus V') \cup E$
- the set of edges $\{(v_i, v_j) : \ v_i, v_j \in V \setminus V'\} \cup \{(v, e) : \ v \in V \setminus V', e \in E, v \text{ is incident to } e\}$,

$$\cup \bigcup_{x \in V'} \{(e_1(x), e_2(x)), (e_1(x), e_3(x)), (e_2(x), e_3(x))\},$$

where $e_1(x), e_2(x), e_3(x)$ are edges, incident to the vertex x.



Definition

The hereditary closure of $\{Q(G) : G \in \mathcal{T}\}$ is denoted by $\mathcal{Q}$, and the hereditary closure of $\{Q^*(G) : G \in \mathcal{T}\}$ is denoted by $\mathcal{Q}^*$.

Result [Alekseev, Korobitsyn, Lozin'04+M.'16]

The classes $\mathcal{T}, \mathcal{D}, \mathcal{Q}, \mathcal{Q}^*$ are boundary for the dominating set problem.

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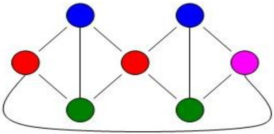
Definition

The class $co(\mathcal{D})$ is the set of complement graphs to graphs in $\mathcal{D}$.

Definition

The chromatic number problem, for a given graph, is to find the minimum k, such that the graph posses a proper vertex k-coloring.



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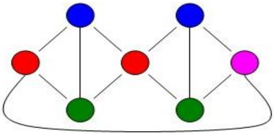

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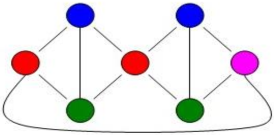


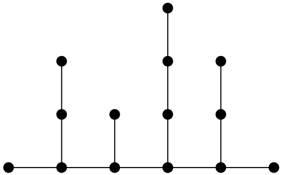
Result [Lozin, Korpelainen, M., Tiskin'11]

The class $co(\mathcal{D})$ is boundary for the chromatic number problem.

Definition

A caterpillar is a graph of maximum vertex degree 3, obtained by coinciding ends of simple paths with vertices of a simple path.

The class $\mathfrak{S}_1$ is the hereditary closure of the set of all caterpillars, the set $\mathfrak{S}_2$ is the hereditary closure of the set of graphs, obtained by inscribing triangles into all degree 3 vertices in all caterpillars.



Result [Lozin, Korpelainen, M.,Tiskin'11]

The classes $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are boundary for the Hamiltonian cycle problem, i.e. the decision problem for a given graph whether it contains a cycle once visiting all its vertices or not.

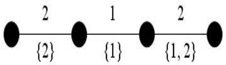
A unique known example of complete descriptions of boundary classes and related issues

Definition

Let G be a graph with an edge set E and $\mathfrak{L} = \{\mathcal{L}(e) | \ e \in E\}$, where every $\mathcal{L}(e)$ is a finite set, consisting of natural numbers. A $\mathfrak{L}$-ranking of G is a coloring of its edges, such that:

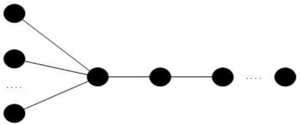
- $c(e) \in \mathcal{L}(e)$, for any edge e;
- if $c(e_1) = c(e_2)$, $e_1 \neq e_2$, then any path, connecting e_1 and e_2, contains an edge e, such that $c(e) > c(e_1)$.

The list edge-ranking problem is, for given G and $\mathfrak{L}$, determine, whether there exists a $\mathfrak{L}$-ranking of G or not.



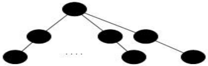
Definition

The class *Comet* is the hereditary closure of the set of all graphs of the form:



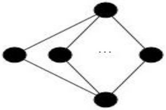
Definition

The class *Star* is the hereditary closure of the set of all graphs of the form:



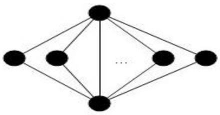
Definition

The class *Bat* is the hereditary closure of the set of all graphs of the form:



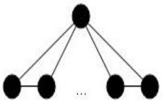
Definition

The class *Comb* is the hereditary closure of the set of all graphs of the form:



Definition

The class *Camomile* is the hereditary closure of the set of all graphs of the form:

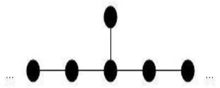


Definition

The class *Clique* is the set of all complete graphs.

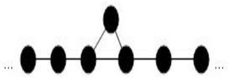
Definition

The class $\tilde{\mathcal{T}}$ is the hereditary closure of the set of all graphs, each connected component of which has the form:



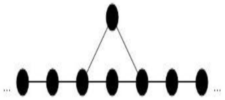
Definition

The class $\tilde{\mathcal{D}}$ is the hereditary closure of the set of all graphs, each connected component of which has the form:



Definition

The class $\hat{\mathcal{T}}$ is the hereditary closure of the set of all graphs, each connected component of which has the form:



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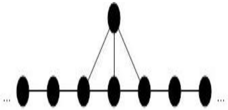

Result [M.'13]

The boundary system for the list edge-ranking problem is constituted by the classes $\mathcal{Star}, \mathcal{Comet}, \mathcal{Bat}, \mathcal{Comb}, \mathcal{Camomile}, \mathcal{Cliques}, \tilde{\mathcal{T}}, \tilde{\mathcal{D}}, \hat{\mathcal{T}}, \hat{\mathcal{D}}$.

Definition

A graph H is called a minor of a graph G if H is obtained from G by deletions of vertices and edges and contractions of edges.

Definition

A class of graphs is minor closed if it is closed under deletion of vertices, edges, and contractions of edges.

Any minor closed class is defined by the set of its obstructions (i.e. forbidden minors), which is always finite, by the well-known Robertson-Seymour theorem.

Definition [M'13]

A class $\mathcal{X}$ is called a minor of a graph class $\mathcal{Y}$ if, for any $H \in \mathcal{X}$, there is a graph $G \in \mathcal{Y}$, for which H is a minor.

Definition [M'13]

A class $\mathcal{X}$ is a strong minor of a class $\mathcal{Y}$ if there exists a polynomial-time algorithm, which, for an arbitrary graph $H \in \mathcal{X}$, computes a graph $G \in \mathcal{Y}$ and a sequence of actions with G, consisting of vertex and edge deletions and edge contractions, which execution results in the graph H.

Definition [M'13]

A hereditary class $\mathcal{X}$ belongs to the family $\mathscr{M}$ iff:

- none of the classes $\mathcal{C}omet, \mathcal{S}tar, \mathcal{B}at$ is a minor of $\mathcal{X}$,
- if a class among $\mathcal{C}omet, \mathcal{S}tar, \mathcal{B}at$ is a minor of $\mathcal{X}$, then it is a strong minor of $\mathcal{X}$.

Result [M'13]

Any minor closed and any finitely defined classes belong to $\mathscr{M}$.

Result [M'13]

The list edge-ranking problem is polynomial-time solvable for a class $\mathcal{X} \in \mathscr{M}$ if and only if none of the classes $\mathcal{B}at, \mathcal{S}tar, \mathcal{C}omet$ is a minor of $\mathcal{X}$; otherwise, it is NP-complete for $\mathcal{X}$.

On difficulties for obtaining complete descriptions of boundary systems for some graph problems

Remark

The set of boundary classes can be too complex for a considering graph problem, and, thus, attempts to give its a complete description seem pointless.

Result [M.'09 and '12]

For any $k \geq 3$, the boundary systems for the vertex and edge k-colourability problems have the continuum cardinality.

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Boundary classes for subsets of the hereditary classes family

Result [Alekseev'04]

The class $\mathcal{T}$ is unique boundary for the independent set problem and the family of monotone classes.

Result [straightforwardly from the Robertson-Seymour theory]

The class $\mathcal{Planar}$ is unique boundary for the independent set problem and the family of minor closed classes.

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The class $\mathcal{T}$ is unique boundary for many graph problems, like the independent set, dominating set, dissociated set problems, and the family of monotone classes. The same is true for $\mathcal{Planar}$ and the family of minor closed classes.

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Result [Lozin'08]

For any d, the only classes $\mathcal{T}$ and $\mathcal{D}$ are boundary for the dominating set problem and the family of hereditary subclasses of $\mathcal{D}eg(d)$.

Result [Korobitsyn'90]

If $Free(\{G\}) \not\supseteq \mathcal{T}, Free(\{G\}) \not\supseteq \mathcal{D}, Free(\{G\}) \not\supseteq \mathcal{Q}$, then the dominating set problem can be solved in polynomial time for $Free(\{G\})$; otherwise, it is NP-complete.

Result [M.'15]

Let a set $\mathcal{Y}$ consist of only graphs, each on at most 5 vertices. Then, the dominating set problem is NP-complete for $Free(\mathcal{Y})$, if it includes at least one of the classes $\mathcal{T}, \mathcal{D}, \mathcal{Q}$; otherwise, it is polynomial-time solvable.

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[7]. Malyshev D. S. Continuum sets of boundary classes of graphs for colorability problems // Diskretnyi Analiz i Issledovanie Operatsii. — 2009. — V. 16, №5. — P. 41–51 [in Russian].

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THANKS FOR YOUR ATTENTION!