A faster algorithm for counting of the integer points in Δ-modular polyhedra *

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Abstract. Let a polytope $\mathcal{P}$ be defined by one of the following ways:

(i) $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \leq b\}$, where $A \in \mathbb{Z}^{(n+m)\times n}$, $b \in \mathbb{Q}^{(n+m)}$, $\operatorname{rank}(A) = n$ and $d := \dim(\mathcal{P}) = n$;

(ii) $\mathcal{P} = \{x \in \mathbb{R}_+^n \colon Ax = b\}$, where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, $\operatorname{rank}(A) = m$ and $d := \dim(\mathcal{P}) = n - m$;

and let all the rank minors of A be bounded by Δ in the absolute values. We show that $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$ can be computed with an algorithm, having the arithmetic complexity bound

$$O\big(\nu(d,m,\Delta) \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)\big),$$

where $\nu(d,m,\Delta)$ is the maximal possible number of vertices in a d-dimensional polytope $\mathcal{P}$, defined by one of the systems above.

Using the obtained result, we have the following arithmetical complexity bounds to compute $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$:

- The bound $O(\frac{d}{m}+1)^m \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is polynomial on d and Δ, for any fixed m. Taking $m = 1$, it gives an $O\big(d^4 \cdot \Delta^4 \cdot \log(\Delta)\big)$-algorithm to compute the number of integer points in a simplex or the number of solutions of the unbounded Subset-Sum problem, where Δ means the maximal weight of an item.
- The bound $O\big(\frac{m}{d}+1\big)^{\frac{d}{2}} \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is polynomial on m and Δ, for any fixed d. The last bound can be used to obtain a faster algorithm for the ILP feasibility problem, when the parameters m and Δ are relatively small. For example, taking $m = O(d)$ and $\Delta = 2^{O(d)}$, the above bound becomes $2^{O(d)}$, which is faster, than the state of the art algorithm, due to [8,9] with the complexity bound $O(d)^d \cdot \operatorname{poly}(\operatorname{size}(A,b))$;
- The bound $O(d)^{3+\frac{d}{2}} \cdot \Delta^{4+d} \cdot \log(\Delta)$ that is polynomial on Δ, for any fixed d. Taking $\Delta = O(1)$, the last bound becomes $O(d)^{3+\frac{d}{2}}$, which again gives a faster algorithm for the ILP feasibility problem, than the state of the art algorithm, due to [8,9].

Keywords: Integer Linear Programming · Short Rational Generating Function · Bounded Sub-Determinants · Multidimensional Knapsack Problem · Subset-Sum Problem · Counting Problem

* The article was prepared within the framework of the Basic Research Program at the National Research University Higher School of Economics (HSE).

1 Introduction

1.1 Basic definitions and notations

Let $A \in \mathbb{Z}^{m \times n}$. We denote by A_{ij} its ij-th element, by A_{i*} its i-th row, and by A_{*j} its j-th column. For subsets $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$, the symbol A_{IJ} denote the sub-matrix of A, which is generated by all the rows with indices in I and all the columns with indices in J. If I or J is replaced by $*$, then all the rows or columns are selected, respectively. Sometimes, we simply write A_I instead of A_{I*} and A_J instead of A_{*J}, if this does not lead to confusion.

The maximum absolute value of entries of a matrix A is denoted by $\|A\|_{\max} = \max_{i,j} |A_{i\,j}|$. The l_p-norm of a vector x is denoted by $\|x\|_p$. The number of non-zero components of a vector x is denoted by $\|x\|_0 = \bigl|\{i\colon x_i \neq 0\}\bigr|$.

Definition 1. *For a matrix $A \in \mathbb{Z}^{m \times n}$, by*

$$\Delta_k(A) = \max \Bigl\{ \bigl|\det(A_{IJ})\bigr| : I \subseteq \{1, \dots, m\},\ J \subseteq \{1, \dots, n\},\ |I| = |J| = k \Bigr\},$$

we denote the maximum absolute value of determinants of all the $k \times k$ sub-matrices of A. By $\Delta_{\gcd}(A, k)$ we denote the greatest common divisor of determinants of all the $k \times k$ sub-matrices of A. Additionally, let $\Delta(A) = \Delta_{\operatorname{rank}(A)}(A)$ and $\Delta_{\gcd}(A) = \Delta_{\gcd}(A, \operatorname{rank}(A))$.

*If $\Delta(A) \leq \Delta$, for some $\Delta > 0$, then A is called Δ-*modular.

Definition 2. *For $A \in \mathbb{Z}^{k \times n}$ and $b \in \mathbb{Z}^k$, we denote*

$$\mathcal{P}(A, b) = \{x \in \mathbb{R}^n \colon Ax \leq b\},$$
$$\mathbf{x}^z = x_1^{z_1} \cdot \ldots \cdot x_n^{z_n}, \quad \text{and} \quad \mathfrak{f}(\mathcal{P}; \mathbf{x}) = \sum_{z \in \mathcal{P} \cap \mathbb{Z}^n} \mathbf{x}^z .$$

1.2 The lattice points counting problem

In this paper, we consider the problem for counting of the polytopes integer points number, which is defined as follows:

Problem 1. Let $\mathcal{P}$ be a rational polytope defined by one of the following ways:

1. The polytope $\mathcal{P}$ is defined by *a system in the canonical form*: $\mathcal{P} = \{x \in \mathbb{R}^n \colon Ax \leq b\}$, where $A \in \mathbb{Z}^{(n+m) \times n}$, $b \in \mathbb{Q}^{n+m}$, and $\dim(\mathcal{P}) = \operatorname{rank}(A) = n$;
2. The polytope $\mathcal{P}$ is defined by *a system in the standard form*: $\mathcal{P} = \{x \in \mathbb{R}^n_{\geq 0} \colon Ax = b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Q}^m$, $\operatorname{rank}(A) = m$, $\dim(\mathcal{P}) = n - m$ and $\Delta_{\gcd}(A) = 1$.

The problem at state is to compute the value of $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$.

The first polynomial-time in a fixed dimension algorithm, which finds the short generating function (its definition will be presented later) for $\mathcal{P}$ and solves Problem 1, was proposed by A. Barvinok in [4]. Further modifications and details were given in [2,3,5,11]. An alternative approach was presented in [17,18].

The paper [15] gives a polynomial-time algorithm for Problem 1 parameterised by m and $\Delta = \Delta(A)$. More precisely, the paper [15] gives an algorithm with the arithmetic complexity bound

$$O\big(T_{SNF}(d) \cdot d^m \cdot d^{\log_2(\Delta)}\big),$$

where $T_{SNF}(d)$ is the complexity of computing the Smith Normal Form for $d \times d$ integer matrices and d is the dimension of the corresponding polytope ($d = n$ for the canonical form and $d = n - m$ for the standard form). We improve the mentioned result from [15] in the following way:

Theorem 1. *Problem 1 can be solved with an algorithm, having the arithmetic complexity bound*

$$O\big(\nu(d, m, \Delta) \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)\big),$$

where $\Delta = \Delta(A)$, $d = \dim(\mathcal{P})$ ($d = n$, for the canonical form, and $d = n - m$, for the standard form) and $\nu(d, m, \Delta)$ is the maximal possible number of vertices in a d-dimensional polytope of Problem 1.

Using this theorem and results of the papers [20,21] that can help to bound the value of $\nu(d, m, \Delta)$, we present new complexity bounds for Problem 1. Additionally, we show how to handle the case of unbounded polyhedra.

Corollary 1. *The arithmetic complexity of an algorithm by Theorem 1 can be bounded with the following relations:*

1. *The bound $O\big(\frac{d}{m} + 1\big)^m \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is polynomial on d and Δ, for any fixed m;*
2. *The bound $O\big(\frac{m}{d} + 1\big)^{\frac{d}{2}} \cdot d^3 \cdot \Delta^4 \cdot \log(\Delta)$ that is on m and Δ, for any fixed d;*
3. *The bound $O(d)^{3+\frac{d}{2}} \cdot \Delta^{4+d} \cdot \log(\Delta)$ that is polynomial on Δ, for any fixed d.*

To handle the case, when $\mathcal{P}$ is an unbounded polyhedron, we need to pay an additional factor of $O(\frac{d}{m} + 1) \cdot d^4$ in the first bound and $O(d^4)$ in the second bound. The third bound stays unchanged.

Proofs of Theorem 1 and Corollary 1 will be given in Section 2 and Subsection 2.4, respectively.

The first bound can be used to count the number of integer points in a simplex or the number of solutions of *the unbounded Subset-Sum problem* $w^\top x = w_0$, $x \in \mathbb{Z}^n_{\geq 0}$. For the both problems, it gives the arithmetic complexity bound $O\big(n^4 \cdot \Delta^4 \cdot \log(\Delta)\big)$, where $\Delta = \|w\|_{\max}$ for the Subset-Sum problem.

The second and third bounds can be used to obtain a faster algorithm for the ILP feasibility problem, when the parameters m and Δ are relatively small. For example, taking $m = O(d)$ and $\Delta = 2^{O(d)}$ in the second bound, it becomes

$2^{O(d)}$, which is faster, than the state of the art algorithm, due to [8,9] (see also [6,13,27], for a bit more general setting) that has the complexity bound $O(d)^d \cdot \operatorname{poly}(\operatorname{size}(A, b))$. Substituting $\Delta = O(1)$ to the third bound, it gives $O(d)^{3+\frac{d}{2}}$, which again is better, than the general case bound $O(d)^d \cdot \operatorname{poly}(\operatorname{size}(A, b))$.

Remark 1. We are interested in development of algorithms that will be polynomial, when we bound some of the parameters d, m, and Δ. Due to [15, Corollary 3], the problem in the standard form can be reduced to the problem in the canonical form maintaining values of m and Δ, see also [14, Lemmas 4 and 5] and [26] for a more general reduction. Hence, in the proofs we will only consider polytopes defined by systems in the canonical form.

Remark 2. To simplify analysis, we assume that $\Delta_{\gcd}(A) = 1$ for ILP problems in the standard form. It can be done without loss of generality, because the original system $Ax = b$, $x \geq \mathbf{0}$ can be polynomially transformed to the equivalent system with $\Delta_{\gcd}(A) = 1$. For the justification see [15, Remark 3].

Good surveys on the related Δ-modular ILP problems and parameterised ILP complexity are given in [12,14,15,25].

1.3 Auxiliary facts from the polyhedral algebra

In this Subsection, we mainly follow to [2,3]. Let $\mathcal{V}$ be a d-dimensional real vector space and $\mathcal{L} \subset \mathcal{V}$ be a lattice.

Definition 3. *Let $\mathcal{A} \subseteq \mathcal{V}$ be a set. The* indicator $[\mathcal{A}]$ *of $\mathcal{A}$ is the function $[\mathcal{A}] \colon \mathcal{V} \to \mathbb{R}$ defined by* $[\mathcal{A}](x) = \begin{cases} 1, \text{ if } x \in \mathcal{A} \\ 0, \text{ if } x \notin \mathcal{A} \end{cases}$.

The algebra of polyhedra $\mathscr{P}(\mathcal{V})$ *is the vector space defined as the span of the indicator functions of all the polyhedra $\mathcal{P} \subset \mathcal{V}$.*

Definition 4. *A linear transformation $\mathcal{T} \colon \mathscr{P}(\mathcal{V}) \to \mathcal{W}$, where $\mathcal{W}$ is a vector space, is called a* valuation. *We consider only* $\mathcal{L}$-valuations *or* lattice valuations *that satisfy*

$$\mathcal{T}([\mathcal{P} + u]) = \mathcal{T}([\mathcal{P}]), \quad \text{for all rational polytopes } \mathcal{P} \text{ and } u \in \mathcal{L},$$

see [22, pp. 933–988], [23].

We are mainly interested in two valuations, the first is the counting valuation $\mathcal{E}([\mathcal{P}]) = |\,\mathcal{P} \cap \mathbb{Z}^d\,|$ and the second valuation $\mathcal{F}([\mathcal{P}])$, which will be significantly used in our paper, is defined by the following theorem, proved by J. Lawrence [19], and, independently, by A. Khovanskii and A. Pukhlikov [24]. We borrowed the formulation from [2, Section 13]:

Theorem 2 ([19,24]). *Let $\mathscr{R}(\mathbb{C}^d)$ be the space of rational functions on $\mathbb{C}^d$ spanned by the functions of the type*

$$\frac{\mathbf{x}^v}{(1 - \mathbf{x}^{u_1}) \dots (1 - \mathbf{x}^{u_d})},$$

where $v \in \mathbb{Z}^d$ *and* $u_i \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$*, for any* $i \in \{1, \dots, d\}$*. Then there exists a linear transformation (a valuation)* $\mathcal{F}\colon \mathscr{P}(\mathbb{Q}^d) \to \mathscr{R}(\mathbb{C}^d)$ *such that the following properties hold:*

1. *Let* $\mathcal{P} \subset \mathbb{R}^d$ *be a non-empty rational polyhedron without lines, and let* $\mathcal{C}$ *be its recession cone. Let* $\mathcal{C}$ *be generated by rays* $w_1, \dots, w_n$*, for some* $w_i \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$*, and let us define*
$$\mathcal{W}_{\mathcal{C}} = \left\{\mathbf{x} \in \mathbb{C}^d \colon |\, \mathbf{x}^{w_i} \,| < 1, \text{ for any } i \in \{1, \dots, n\}\right\}.$$
Then, $\mathcal{W}_{\mathcal{C}}$ *is a non-empty open set and, for all* $\mathbf{x} \in \mathcal{W}_{\mathcal{C}}$*, the series*
$$\mathfrak{f}(\mathcal{P}; \mathbf{x}) = \sum_{z \in \mathcal{P} \,\cap\, \mathbb{Z}^d} \mathbf{x}^z$$
converges absolutely and uniformly on compact subsets of $\mathcal{W}_{\mathcal{C}}$ *to the function* $f(\mathcal{P}; \mathbf{x}) = \mathcal{F}([\mathcal{P}]) \in \mathscr{R}(\mathbb{C}^d)$*.*
2. *If* P *contains a line, then* $f(\mathcal{P}; \mathbf{x}) = 0$*.*

If $\mathcal{P}$ is a rational polyhedron, then $f(\mathcal{P}; \mathbf{x})$ is called its *short rational generating function.*

Definition 5. *Let* $\mathcal{P} \subset \mathcal{V}$ *be a non-empty polyhedron, and let* $v \in \mathcal{P}$ *be a point. We define the* tangent cone *of* $\mathcal{P}$ *at* v *by*
$$\operatorname{tcone}(\mathcal{P}, v) = \left\{v + y \colon v + \varepsilon y \in \mathcal{P}, \quad \text{for some } \varepsilon > 0\right\}.$$
If an n*-dimensional polyhedron* $\mathcal{P}$ *is defined by a system* $Ax \le b$ *and* $\mathcal{P}$ *contains no lines, then, for any* $v \in \operatorname{vert}(\mathcal{P})$*, it holds*
$$\operatorname{tcone}(\mathcal{P}, v) = \{x \in \mathcal{V} \colon A_{\mathcal{J}(v)*} x \le b_{\mathcal{J}(v)}\}, \quad \text{where } \mathcal{J}(v) = \{j \colon A_{j*} v = b_j\}.$$

It is widely known that a slight perturbation in the right-hand sides of a system $Ax \le b$ can transform the polyhedron $\mathcal{P}(A, b)$ to a simple one. Here, we need an algorithmic version of this fact, presented in the following technical theorem.

Theorem 3. *Let* $A \in \mathbb{Z}^{k \times n}$, $\operatorname{rank}(A) = n \le k$, $b \in \mathbb{Q}^k$, $\gamma = \max\{\|A\|_{\max}, \|b\|_\infty\}$, *and* $\mathcal{P} = \mathcal{P}(A, b)$ *be the* n*-dimensional polyhedron.*

Then, for $1/\varepsilon = 1 + 2n \cdot n^{\lceil n/2 \rceil} \cdot \gamma^n$ *and the vector* $t \in \mathbb{Q}^k$*, with* $t_i = \varepsilon^{i-1}$*, the polyhedron* $\mathcal{P}' = \mathcal{P}(A, b + t)$ *is simple.*

Proof. Let us suppose by the contrary that there exists a vertex v of $\mathcal{P}'$ and a set of indices $\mathcal{J}$ such that $A_{\mathcal{J}} v = (b+t)_{\mathcal{J}}$, $|\,\mathcal{J}\,| = n+1$ and $\operatorname{rank}(A_{\mathcal{J}}) = n$. The last is possible iff $\det(M) = 0$, where $M = \begin{pmatrix} A_{\mathcal{J}} \; (b+t)_{\mathcal{J}} \end{pmatrix}$. Note that $M = B + D$, where $B = \begin{pmatrix} A_{\mathcal{J}} \; b_{\mathcal{J}} \end{pmatrix}$ and $D = \begin{pmatrix} \mathbf{0}_{(n+1) \times n} \; t_{\mathcal{J}} \end{pmatrix}$. We have,
$$\det(M) = \det(B) + \sum_{i=1}^{n+1} \det(B[i, t_{\mathcal{J}}]) =$$
$$= \det(B) + \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (-1)^{i+j} \cdot (t_{\mathcal{J}})_j \cdot \det(B_{\mathcal{J} \setminus \{j\} \, \mathcal{I} \setminus \{i\}}),$$

where $\mathcal{I} = \{1, \dots, n+1\}$ and $B[i, t_{\mathcal{J}}]$ is the matrix induced by the substitution of the column $t_{\mathcal{J}}$ instead of i-th column of B.

Let us assume that $(t_{\mathcal{J}})_j = \varepsilon^{d_j}$, for $j \in \mathcal{I}$, where $d_j \in \mathbb{Z}$ and $0 \le d_1 < d_2 < \dots < d_{n+1} \le k-1$. Consequently, the condition $\det(M) = 0$ is equivalent to the following condition:

$$\det(B) + \sum_{j=1}^{n+1} \varepsilon^{d_j} \cdot \left(\sum_{i=1}^{n} (-1)^{i+j} \cdot \det(B_{\mathcal{J} \setminus \{j\}\, \mathcal{I} \setminus \{i\}}) \right) = 0. \qquad (1)$$

Note that the polynomial (1) is not zero. Definitely, since $\operatorname{rank}(A_{\mathcal{J}}) = n$, we can assume that the first n rows of $A_{\mathcal{J}}$ are linearly independent. Consequently, there exists a unique vector $y \in \mathbb{Q}^n_{\neq 0}$ such that the last row of $A_{\mathcal{J}}$ is a linear combination of the first rows with the coefficients vector y. Since $\forall \varepsilon \colon \det(M) = 0$, we have $\binom{y}{1}^\top M = \mathbf{0}$ and, consequently, $\binom{y}{1}^\top (b_{\mathcal{J}} + t) = \mathbf{0}$. But, the last may hold only for a finite number of ε. That is the contradiction.

As a corollary of the Rouché's theorem, we have that $|\varepsilon^*| \ge \frac{1}{1+\alpha_{\max}/\beta} = \frac{\beta}{\beta+\alpha_{\max}}$, where ε^* is any root of (1), $\alpha_{\max}$ is the maximal absolute value of the coefficients, and β is the absolute value of the non-zero coefficient with a minimal index.

Finally, $1/|\varepsilon^*| \le 2\alpha_{\max} \le 2n \cdot n^{n/2} \cdot \gamma^n$, which contradicts to the Theorem's condition on ε.

2 Proof of Theorem 1

2.1 A recurrent formula for the generating function of a group polyhedron

Let $\mathcal{G}$ be a finite Abelian group and $g_1, \dots, g_n \in \mathcal{G}$. Let, additionally, $r_i = \left|\langle g_i \rangle\right|$ be the order of g_i, for $i \in \{1, \dots, n\}$, and $r_{\max} = \max_i r_i$. For $g_0 \in \mathcal{G}$ and $k \in \{1, \dots, n\}$, let $\mathcal{P}_{\mathcal{G}}(k, g_0)$ be the polyhedron induced by the convex hull of solutions of the following system:

$$\begin{cases} \sum\limits_{i=1}^{k} x_i g_i = g_0 \\ x \in \mathbb{Z}^k_{\ge 0}\,. \end{cases} \qquad (2)$$

Let us consider the formal power series $\mathfrak{f}_k(g_0; \mathbf{x}) = \sum_{z \in \mathcal{P}_{\mathcal{G}}(k, g_0) \cap \mathbb{Z}^k} \mathbf{x}^z$.

For $k = 1$, we clearly have

$$\mathfrak{f}_1(g_0; \mathbf{x}) = \frac{x_1^s}{1 - x_1^{r_1}}, \quad \text{where } s = \min\{x_1 \in \mathbb{Z}_{\ge 0} \colon x_1 g_1 = g_0\}.$$

If such s does not exist, then we put $\mathfrak{f}_1(g_0; \mathbf{x}) = 0$.

Note that, for any value of $x_k \in \mathbb{Z}_{\geq 0}$, the system (2) can be rewritten as

$$\begin{cases} \sum\limits_{i=1}^{k-1} x_i g_i = g_0 - x_k g_k \\ x \in \mathbb{Z}_{\geq 0}^{k-1} \,. \end{cases}$$

Hence, for $k \geq 1$, we have

$$\mathfrak{f}_k(g_0; \mathbf{x}) = \\ = \frac{\mathfrak{f}_{k-1}(g_0;\mathbf{x}) + x_k \cdot \mathfrak{f}_{k-1}(g_0 - g_k;\mathbf{x}) + \dots + x_k^{r_k - 1} \cdot \mathfrak{f}_{k-1}(g_0 - g_k \cdot (r_k - 1);\mathbf{x})}{1 - x_k^{r_k}} = \\ = \frac{1}{1 - x_k^{r_k}} \cdot \sum_{i=0}^{r_k - 1} x_k^i \cdot \mathfrak{f}_{k-1}(g_0 - i \cdot g_k; \mathbf{x}). \quad (3)$$

$$\text{Consequently,} \quad \mathfrak{f}_k(g_0;\mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1,\dots,i_k} x_1^{i_1} \dots x_k^{i_k}}{(1 - x_1^{r_1})(1 - x_2^{r_2}) \dots (1 - x_k^{r_k})}, \quad (4)$$

where the nominator is a polynomial with coefficients $\epsilon_{i_1,\dots,i_k} \in \{0, 1\}$ and degree at most $(r_1 - 1) \dots (r_k - 1)$. Additionally, the formal power series $\mathfrak{f}_k(g_0;\mathbf{x})$ converges absolutely to the given rational function if $|x_i^{r_i}| < 1$, for each $i \in \{1, \dots, k\}$.

2.2 Simple Δ-modular polyhedral cone and its generating function

Let $A \in \mathbb{Z}^{n \times n}$, $b \in \mathbb{Z}^n$, $\Delta = |\det(A)| > 0$, $\mathcal{P} = \mathcal{P}(A, b)$, and let us consider the formal power series

$$\mathfrak{f}(\mathcal{P};\mathbf{x}) = \sum_{z \in \mathcal{P} \cap \mathbb{Z}^n} \mathbf{x}^z \,.$$

Let $A = P^{-1} S Q^{-1}$ and $\sigma = S_{nn} = \Delta / \Delta_{\gcd}(A, n-1)$, where $S \in \mathbb{Z}^{n \times n}$ is the SNF of A and $P, Q \in \mathbb{Z}^{n \times n}$ are unimodular matrices. After the unimodular map $x = Q x'$ and introducing slack variables y, the system $Ax \leq b$ becomes

$$\begin{cases} Sx + Py = Pb \\ x \in \mathbb{Z}^n \\ y \in \mathbb{Z}_{\geq 0}^n \,. \end{cases}$$

Since P is unimodular, the last system is equivalent to the system

$$\begin{cases} Py = Pb \pmod{S} \\ y \in \mathbb{Z}_{\geq 0}^n \,. \end{cases} \quad (5)$$

Note that points of $\mathcal{P} \cap \mathbb{Z}^n$ and the system (5) are connected by the bijective map $x = A^{-1}(b - y)$.

The system (5) can be interpreted as a group system (2), where $\mathcal{G} = \mathbb{Z}^n \bmod S$ with an addition modulo S, $k = n$, $g_0 = Pb \bmod S$ and $g_i = P_{*i} \bmod S$, for $i \in \{1, \dots, n\}$. Clearly, $|\,\mathcal{G}\,| = |\det(S)| = \Delta$ and $r_{\max} \le \sigma$.

Following to the previous Subsection, for $k \in \{1, \dots, n\}$ and $g_0 \in \mathcal{G}$, let $\mathcal{M}_k(g_0)$ be the solutions set of the system

$$\begin{cases} \sum\limits_{i=1}^{k} y_i g_i = g_0 \\ y \in \mathbb{Z}^k_{\ge 0}, \end{cases} \qquad \text{and} \quad \mathfrak{f}_k(g_0; \mathbf{x}) = \sum_{y \in \mathcal{M}_k(g_0)} \mathbf{x}^{-\sum\limits_{i=1}^{k} h_i y_i},$$

where h_i is the i-th column of the matrix $A^* = \Delta \cdot A^{-1}$.

Note that

$$\mathfrak{f}(\mathcal{P}; \mathbf{x}) = \sum_{z \in \mathcal{P} \cap \mathbb{Z}^n} \mathbf{x}^z = \sum_{y \in \mathcal{M}_n(Pb \bmod S)} \mathbf{x}^{A^{-1}(b-y)} =$$
$$= \mathbf{x}^{A^{-1}b} \cdot \sum_{y \in \mathcal{M}_n(Pb \bmod S)} \mathbf{x}^{-\frac{1}{\Delta} A^* y} = \mathbf{x}^{A^{-1}b} \cdot \mathfrak{f}_n(Pb \bmod S; \mathbf{x}^{\frac{1}{\Delta}}). \quad (6)$$

Due to (3) and (4), for $k = 1$, we have

$$\mathfrak{f}_1(g_0; \mathbf{x}) = \frac{\mathbf{x}^{-s h_1}}{1 - \mathbf{x}^{-r_1 h_1}}, \quad \text{where } s = \min\{y_1 \in \mathbb{Z}_{\ge 0} \colon y_1 g_1 = g_0\}. \quad (7)$$

For $k \ge 2$, we have

$$\mathfrak{f}_k(g_0; \mathbf{x}) = \frac{1}{1 - \mathbf{x}^{-r_k h_k}} \cdot \sum_{i=0}^{r_k - 1} \mathbf{x}^{-i h_k} \cdot \mathfrak{f}_{k-1}(g_0 - i \cdot g_k; \mathbf{x}) \text{ and} \quad (8)$$

$$\mathfrak{f}_k(g_0; \mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1,\dots,i_k} \mathbf{x}^{-(i_1 h_1 + \dots + i_k h_k)}}{(1 - \mathbf{x}^{-r_1 h_1})(1 - \mathbf{x}^{-r_2 h_2}) \dots (1 - \mathbf{x}^{-r_k h_k})}, \quad (9)$$

where the nominator is a Laurent polynomial with coefficients $\epsilon_{i_1,\dots,i_k} \in \{0, 1\}$. Clearly, the power series $\mathfrak{f}_k(g_0; \mathbf{x})$ converges absolutely to the given function if $|\,\mathbf{x}^{-r_i h_i}\,| < 1$, for each $i \in \{1, \dots, k\}$.

Due to the formulae (9) and (6), we have

$$\mathfrak{f}(\mathcal{P}; \mathbf{x}) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_n=0}^{r_n-1} \epsilon_{i_1,\dots,i_n} \mathbf{x}^{\frac{1}{\Delta} A^*(b - (i_1,\dots,i_n)^\top)}}{\left(1 - \mathbf{x}^{-\frac{r_1}{\Delta} h_1}\right)\left(1 - \mathbf{x}^{-\frac{r_2}{\Delta} h_2}\right) \dots \left(1 - \mathbf{x}^{-\frac{r_n}{\Delta} h_n}\right)}. \quad (10)$$

Note that $\frac{r_i}{\Delta} h_i$ is an integer vector, for any $i \in \{1, \dots, n\}$, and $\frac{1}{\Delta} A^*(b - (i_1, \dots, i_n)^\top)$ is an integer vector, for any $(i_1, \dots, i_n)$, such that $\epsilon_{i_1,\dots,i_n} \ne 0$. Indeed, by definition of r_i, we have $r_i P_{*i} \equiv \mathbf{0} \pmod S$, so $\frac{r_i}{\Delta} h_i = (r_i A^{-1})_{*i} = (Q S^{-1} P r_i)_{*i}$, which is an integer vector. Vectors $(i_1, \dots, i_n)^\top$ correspond to solutions y of the system (5), and $\frac{1}{\Delta} A^*(b - (i_1, \dots, i_n)^\top) = A^{-1}(b - y)$ is an integer vector.

Additionally, note that the vectors $-\frac{r_i}{\Delta} h_i$ represent extreme rays of the recession cone of $\mathcal{P}$.

Let $c \in \mathbb{Z}^n$ be chosen, such that $(c^\top A^*)_i \neq 0$, for any i. Let us consider the exponential sum

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \sum_{y \in \mathcal{M}_k(g_0)} e^{-\tau \cdot \langle c, \sum_{i=1}^k h_i y_i \rangle}$$

that is induced by $\mathfrak{f}_k(g_0; \mathbf{x})$, substituting $x_i = e^{\tau \cdot c_i}$.

The formulae (7), (8), and (9) become

$$\hat{\mathfrak{f}}_1(g_0; \tau) = \frac{e^{-\langle c, s h_1 \rangle \cdot \tau}}{1 - e^{-\langle c, r_1 h_1 \rangle \cdot \tau}}, \tag{11}$$

$$\hat{\mathfrak{f}}_k(g_0; \mathbf{x}) = \frac{1}{1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}} \cdot \sum_{i=0}^{r_k - 1} e^{-\langle c, i h_k \rangle \cdot \tau} \cdot \hat{\mathfrak{f}}_{k-1}(g_0 - i \cdot g_k; \tau), \tag{12}$$

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \frac{\sum\limits_{i_1=0}^{r_1-1} \dots \sum\limits_{i_k=0}^{r_k-1} \epsilon_{i_1, \dots, i_k} e^{-\langle c, i_1 h_1 + \dots + i_k h_k \rangle \cdot \tau}}{\left(1 - e^{-\langle c, r_1 h_1 \rangle \cdot \tau}\right)\left(1 - e^{-\langle c, r_2 h_2 \rangle \cdot \tau}\right) \dots \left(1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}\right)}. \tag{13}$$

Let $\chi = \max\limits_{i \in \{1, \dots, n\}} \{|\langle c, h_i \rangle|\}$. Since $\langle c, h_i \rangle \in \mathbb{Z}_{\neq 0}$, for each i, the number of therms $e^{-\langle c, \cdot \rangle \cdot \tau}$ is bounded by $1 + 2 \cdot k \cdot r_{\max} \cdot \chi \leq 1 + 2 \cdot k \cdot \sigma \cdot \chi$. So, after combining similar therms, the nominator's length becomes $O(k \cdot \sigma \cdot \chi)$.

In other words, there exist coefficients $\epsilon_i \in \mathbb{Z}_{\geq 0}$, such that

$$\hat{\mathfrak{f}}_k(g_0; \tau) = \frac{\sum\limits_{i=-k \cdot \sigma \cdot \chi}^{k \cdot \sigma \cdot \chi} \epsilon_i \cdot e^{-i \cdot \tau}}{\left(1 - e^{-\langle c, r_1 \cdot h_1 \rangle \tau}\right)\left(1 - e^{-\langle c, r_2 h_2 \rangle \cdot \tau}\right) \dots \left(1 - e^{-\langle c, r_k h_k \rangle \cdot \tau}\right)}. \tag{14}$$

Let us discuss the group-operations complexity issues to find the representation (14) of $\hat{\mathfrak{f}}_k(g_0; \tau)$, for any $k \in \{1, \dots, n\}$ and $g_0 \in \mathcal{G}$.

Clearly, to find the desired representation of $\hat{\mathfrak{f}}_1(g_0; \tau)$, for all $g_0 \in \mathcal{G}$, we need $r_1 \cdot \Delta$ group operations.

Fix $g_0 \in \mathcal{G}$ and $k \in \{1, \dots, n\}$. To find $\hat{\mathfrak{f}}_k(g_0; \tau)$, for $k \geq 2$, we can use the formula (12). Each nominator of the therm $e^{-\langle c, i h_k \rangle \cdot \tau} \cdot \hat{\mathfrak{f}}_{k-1}(g_0 - i g_k; \tau)$ contains at most $1 + 2 \cdot (k-1) \cdot \sigma \cdot \chi$ non-zero therms of the type $\epsilon \cdot e^{-\langle c, \cdot \rangle \cdot \tau}$. Hence, the summation can be done with $O(k \cdot \sigma^2 \cdot \chi)$ group operations. Consequently, the total group-operations complexity can be expressed by the formula

$$O(\Delta \cdot n^2 \cdot \sigma^2 \cdot \chi).$$

Finally, since the diagonal matrix S can have at most $\log_2(\Delta)$ therms that are not equal to 1, the arithmetic complexity of one group operation is $O(\log(\Delta))$. Hence, the total arithmetic complexity is

$$O\big(\Delta \cdot \log(\Delta) \cdot n^2 \cdot \sigma^2 \cdot \chi\big).$$

Finally, let us show how to find the exponential form

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \sum_{z \in \mathcal{P} \cap \mathbb{Z}^n} e^{\langle c, z\rangle \cdot \tau}$$

of the power series $\mathfrak{f}(\mathcal{P};\mathbf{x})$ induced by the map $x_i = e^{c_i \cdot \tau}$.

Due to the formula (6), we have

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = e^{\langle c, A^{-1} b\rangle \cdot \tau} \cdot \hat{\mathfrak{f}}_n(Pb \bmod S; \frac{\tau}{\Delta}).$$

Due to the last formula and the formulae (10) and (14), we have

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i \cdot e^{\frac{1}{\Delta}(\langle c, A^* b\rangle - i)\cdot\tau}}{\left(1 - e^{-\langle c, \frac{r_1}{\Delta}\cdot h_1\rangle\cdot\tau}\right)\left(1 - e^{-\langle c, \frac{r_2}{\Delta}\cdot h_2\rangle\cdot\tau}\right)\dots\left(1 - e^{-\langle c, \frac{r_n}{\Delta}\cdot h_n\rangle\cdot\tau}\right)}.$$

Again, due to (10), we have $\langle c, \frac{r_i}{\Delta} h_i\rangle \in \mathbb{Z}_{\neq 0}$, for any $i \in \{1, \dots, n\}$, and $\frac{1}{\Delta}(\langle c, A^* b\rangle - i) \in \mathbb{Z}$, for any i, such that $\epsilon_i > 0$.

We have proven the following:

Theorem 4. *Let $A \in \mathbb{Z}^{n\times n}$, $b \in \mathbb{Z}^n$, $\Delta = |\det(A)| > 0$, and $\mathcal{P} = \mathcal{P}(A, b)$. Let, additionally, $\sigma = S_{nn}$, where S is the SNF of A, and $\chi = \max\limits_{i\in\{1,\dots,n\}} \big\{|\langle c, h_i\rangle|\big\}$, where h_i is the i-th column of $A^* = \Delta \cdot A^{-1}$.*

Then, the formal exponential series $\hat{\mathfrak{f}}(\mathcal{P};\tau)$ can be represented as

$$\hat{\mathfrak{f}}(\mathcal{P};\tau) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i \cdot e^{\alpha_i \cdot\tau}}{\left(1 - e^{\beta_1\cdot\tau}\right)\left(1 - e^{\beta_2\cdot\tau}\right)\dots\left(1 - e^{\beta_n\cdot\tau}\right)},$$

where $\epsilon_i \in \mathbb{Z}_{\geq 0}$, $\beta_i \in \mathbb{Z}_{\neq 0}$, and $\alpha_i \in \mathbb{Z}$.

This representation can be found with an algorithm having the arithmetic complexity bound $O\big(\Delta \cdot \log(\Delta) \cdot n^2 \cdot \sigma^2 \cdot \chi\big)$.

2.3 Handling the general case

Following to Remark 1, we will only work with systems in the canonical form. Let $A \in \mathbb{Z}^{(n+m)\times n}$, $b \in \mathbb{Q}^{n+m}$, $\operatorname{rank}(A) = n$, and $\Delta = \Delta(A)$. Let us consider the polytope $\mathcal{P} = \mathcal{P}(A, b)$.

Let us choose $\gamma = \max\{\|A\|_{\max}, \|b\|_\infty\}$, $\beta = \min\limits_{i\in\{1,\dots,n+m\}}\{\lceil b_i \rceil - b_i \colon b_i \notin \mathbb{Z}\}$, and $\varepsilon = \min\{\beta/2,\, (1 + 2n \cdot n^{\lceil n/2\rceil} \cdot \gamma)^{-1}\}$. Then, by Theorem 3, the polytope $\mathcal{P}' = \mathcal{P}(A, b+t)$ is simple, where the vector t is chosen, such that $t_i = \varepsilon^{i-1}$, for $i \in \{1, \dots, n+m\}$. By the construction, $\mathcal{P} \cap \mathbb{Z}^n = \mathcal{P}' \cap \mathbb{Z}^n$. From this moment, we assume that $\mathcal{P} = \mathcal{P}(A, b)$ is a simple polytope.

Using Definition 5, the Brion's Theorem gives:

$$[\mathcal{P}] = \sum_{v \in \operatorname{vert}(\mathcal{P})} \bigl[\operatorname{tcone}(\mathcal{P}, v)\bigr] = \sum_{v \in \operatorname{vert}(\mathcal{P})} \bigl[\mathcal{P}(A_{\mathcal{J}(v)}, b_{\mathcal{J}(v)})\bigr] \quad \textit{modulo polyhedra with lines}.$$

Due to the seminal work [1], all vertices of the simple polyhedra $\mathcal{P}$ can be enumerated with $O\bigl((m+n) \cdot n \cdot |\operatorname{vert}(\mathcal{P})|\bigr)$ arithmetic operations.

Denote $f(\mathcal{P}; \mathbf{x}) = \mathcal{F}([\mathcal{P}]) \in \mathscr{R}(\mathbb{Q}^n)$, for any rational polyhedra $\mathcal{P}$, where $\mathcal{F}$ is the evaluation considered in Theorem 2.

Note that $f(\mathcal{P}(B,u); \mathbf{x}) = f(\mathcal{P}(B, \lfloor u \rfloor); \mathbf{x})$, for any $B \in \mathbb{Q}^{n \times n}$ and $u \in \mathbb{Q}^n$. So, due to Theorem 2, we can write

$$f(\mathcal{P}; \mathbf{x}) = \sum_{v \in \operatorname{vert}(\mathcal{P})} f\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \mathbf{x}\Bigr).$$

Due to results of the previous Subsection, each therm $f\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \mathbf{x}\Bigr)$ has the form (10).

To find the value of $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = \lim\limits_{\mathbf{x} \to \mathbf{1}} f(\mathcal{P}; \mathbf{x})$, we follow to Chapters 13 and 14 of [2]. Let us choose $c \in \mathbb{Z}^n$, such that any element of the row-vector $c^\top (A_{\mathcal{J}(v)})^{-1}$ is non-zero, for each $v \in \operatorname{vert}(\mathcal{P})$. Substituting $x_i = e^{c_i \cdot \tau}$, let us consider the exponential function

$$\hat{f}(\mathcal{P}; \tau) = \sum_{v \in \operatorname{vert}(\mathcal{P})} \hat{f}\Bigl(\mathcal{P}\bigl(A_{\mathcal{J}(v)}, \lfloor b_{\mathcal{J}(v)} \rfloor\bigr); \tau\Bigr).$$

Due to [2, Chapter 14], the value $|\,\mathcal{P} \cap \mathbb{Z}^n\,|$ is a constant therm in the Tailor series of the function $\hat{f}(\mathcal{P}; \tau)$, so we just need to compute it.

Let us fix some therm $\hat{f}\bigl(\mathcal{P}(B,u); \tau\bigr)$ of the previous formula. Due to Theorem 4, it can be represented as

$$\hat{f}\bigl(\mathcal{P}(B,u); \tau\bigr) = \frac{\sum\limits_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \epsilon_i \cdot e^{\alpha_i \cdot \tau}}{\bigl(1 - e^{\beta_1 \cdot \tau}\bigr)\bigl(1 - e^{\beta_2 \cdot \tau}\bigr) \dots \bigl(1 - e^{\beta_n \cdot \tau}\bigr)},$$

where $\epsilon_i \in \mathbb{Z}_{\geq 0}$, $\beta_i \in \mathbb{Z}_{\neq 0}$, and $\alpha_i \in \mathbb{Z}$.

Again, due to [2, Chapter 14], we can see that the constant therm in Tailor series for $\hat{f}\bigl(\mathcal{P}(B,u); \tau\bigr)$ is exactly

$$\sum_{i=-n\cdot\sigma\cdot\chi}^{n\cdot\sigma\cdot\chi} \frac{\epsilon_i}{\beta_1 \dots \beta_n} \sum_{j=0} \frac{\alpha_i^j}{j!} \cdot \operatorname{td}_{n-j}(\beta_1, \dots, \beta_n), \tag{15}$$

where $\operatorname{td}_j(\beta_1, \dots, \beta_n)$ is a homogeneous polynomial of degree j, called the *j-th Todd polynomial* on $\beta_1, \dots, \beta_n$. Due to [10, Theorem 7.2.8, p. 137], the value of $\operatorname{td}_j(\beta_1, \dots, \beta_n)$ can be computed with an algorithm that is polynomial on j, n, and the bit-encoding length of $\beta_1, \dots, \beta_n$.

Since $\sigma \leq \Delta$, due to Theorem 4, the total arithmetic complexity to find the value of (15) can be bounded by $O\big(\Delta^3 \cdot \log(\Delta) \cdot n^2 \cdot \chi\big)$.

The constant therm in Tailor series for the complete function $\hat{f}(\mathcal{P};\tau)$ can be found just by summation. It gives the arithmetic complexity bound

$$O\big(\nu(n,m,\Delta) \cdot n^2 \cdot \Delta^3 \cdot \log(\Delta) \cdot \chi\big).$$

Finally, we choose $c^\top$ as the sum of rows of some non-degenerate $n \times n$ sub-matrix of A. Note that elements of the matrix $A \cdot A^*_{\mathcal{J}(v)}$ are included to the set of all $n \times n$ sub-determinants of A, where $A^*_{\mathcal{J}(v)} = \Delta \cdot A^{-1}_{\mathcal{J}(v)}$, for all $v \in \operatorname{vert}(\mathcal{P})$. Hence, $\chi \leq n\Delta$, and the total arithmetic complexity bound becomes

$$O\big(\nu(n,m,\Delta) \cdot n^3 \cdot \Delta^4 \cdot \log(\Delta)\big). \quad \text{It finishes the proof of Theorem 1.}$$

2.4 Proof of Corollary 1

The presented complexity bounds follow by the different ways to estimate the value $\nu(m,n,\Delta)$. The first bound trivially follows from the inequalities $\nu(m,n,\Delta) \leq \binom{n+m}{n} = \binom{n+m}{m} \lesssim \frac{e^m \cdot (n+m)^m}{m^m} = O\big(\frac{n}{m}+1\big)^m$.

To obtain the second bound, we refer to the seminal result, due to P. McMullen [21]. Together with the formula from [16, Section 4.7] for the number of facets of a cyclic polytope, it follows that the maximal number of vertices in an n-dimensional polyhedron with k facets is bounded by

$$\xi(n,k) = \begin{cases} \frac{k}{k-s}\binom{k-s}{s}, \text{ for } n = 2s \\ 2\binom{k-s-1}{s}, \text{ for } n = 2s+1 \end{cases} = O\left(\frac{k}{n}\right)^{n/2}.$$

Clearly, $\nu(m,n,\Delta) \leq \xi(n, n+m)$, and $\nu(m,n,\Delta) = O\big(\frac{n+m}{n}\big)^{\frac{n}{2}}$. So, the second bound holds.

Due to [20], we can assume that $n + m = O(n^2 \cdot \Delta^2)$. Substituting the last formula to the second bound, we obtain $\nu(m,n,\Delta) = O(n^{\frac{n}{2}} \cdot \Delta^n)$, and the third bound holds.

Finally, let us show how to handle the case, when $\mathcal{P}$ is an unbounded n-dimensional polyhedron. Clearly, we need to distinguish between two possibilities: $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = 0$ and $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = \infty$. Let us choose any vertex v of $\mathcal{P}$ and consider a set of indices $\mathcal{J}$, such that $|\,\mathcal{J}\,| = n$, $A_{\mathcal{J}} v = b_{\mathcal{J}}$ and $\operatorname{rank}(A_{\mathcal{J}}) = n$. For the first and second bounds, we add a new inequality $c^\top x \leq c_0$ to the system $Ax \leq b$, where $c^\top = \sum_{i=1}^n (A_{\mathcal{J}})_{i*}$ and $c_0 = c^\top v + \|c\|_1 \cdot n\Delta + 1$. Let $A'x \leq b'$ be the new system. Due to [7], $|\,\mathcal{P} \cap \mathbb{Z}^n\,| = 0$ iff $|\,\mathcal{P}(A',b') \cap \mathbb{Z}^n\,| = 0$. Since $\mathcal{P}(A',b')$ is a polytope and $\Delta(A') \leq n\Delta$, we just need to add an additional multiplicative factor of $O(\frac{d}{m}+1) \cdot n^4$ to the first bound and $O(n^4)$ to the second bound.

To deal with third bound, we just need to add additional inequalities $A_{\mathcal{J}} x \geq b_{\mathcal{J}} - \|A_{\mathcal{J}}\|_{\max} \cdot n^2 \Delta \cdot \mathbf{1}$ to the system $Ax \leq b$. The polyhedron becomes bounded and the sub-determinants stay unchanged, and we follow to the original scenario.

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