Subexponential size $\mathbb{R}P^n$

Sergey Avvakumov [1]
joint work with Karim Adiprasito[1,2] and Roman Karasev [3,4]

[1]University of Copenhagen
[2]Einstein Institute of Mathematics, Hebrew University of Jerusalem
[3]Moscow Institute of Physics and Technology
[4]Institute for Information Transmission Problems RAS

Combinatorics and Geometry Days III, December 2020

Sizes of manifold's triangulations

Denote by $|M|$ the minimum number of vertices in a triangulation of a manifold M. For a family of manifolds M_n it is interesting to know the asymptotic behavior of $|M_n|$.
A trivial example

$$|S^n| = n + 2.$$

Best current bounds for other basic families

$$O(n^2) \leq |(S^1)^n| \leq 2^{n+1} - 1,$$
$$O(n^2) \leq |\mathbb{R}P^n| \leq c^n,$$
$$O(n^2) \leq |\mathbb{C}P^n| \leq ?.$$

Inductive constructions of $\mathbb{R}P^n$

A very simple inductive construction gives

$$|\mathbb{R}P^n| \leq 3 \cdot 2^{n-1}.$$

Using Minkowski sum construction one can prove

$$|\mathbb{R}P^{kn}| \leq |\mathbb{R}P^n|^k.$$

There is also a very nice inductive construction by Venturello and Zheng with

$$|\mathbb{R}P^{n+2}| \leq |\mathbb{R}P^{n+1}| + |\mathbb{R}P^n|.$$

Our result

Theorem (Adiprasito, A., Karasev)

For all positive integers n, there exists a triangulation of $\mathbb{R}P^{n-1}$ with at most $e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}$ vertices.

So,

$$O(n^2) \leq |\mathbb{R}P^n| \leq e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}.$$

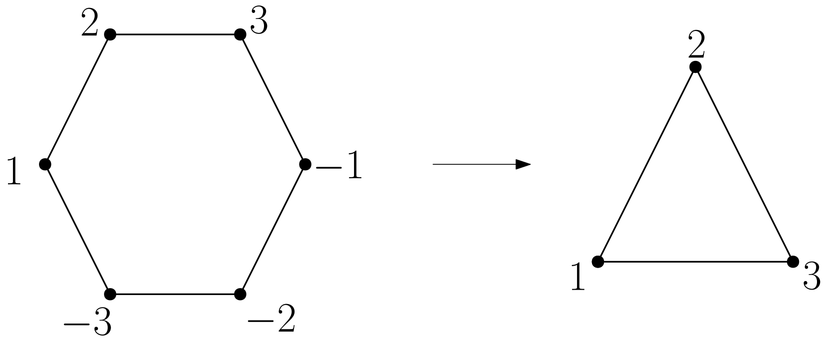
$\mathbb{R}P^n$ is a symmetric sphere of diameter at least 3

Consider the universal cover of $\mathbb{R}P^n$. It is a triangulation of S^n which is

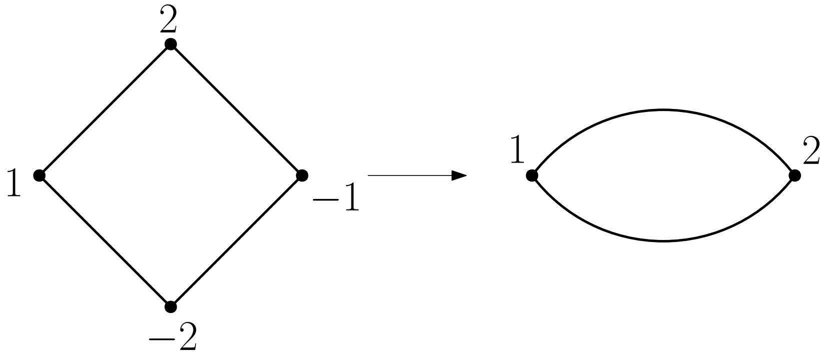
- symmetric,
- the closed stars of any two opposite vertices are disjoint, equivalently
- the distance (number of edges) between any two opposite vertices is at least 3.

In the other direction, the quotient of any triangulation of S^n with the properties above is a triangulation of $\mathbb{R}P^n$.

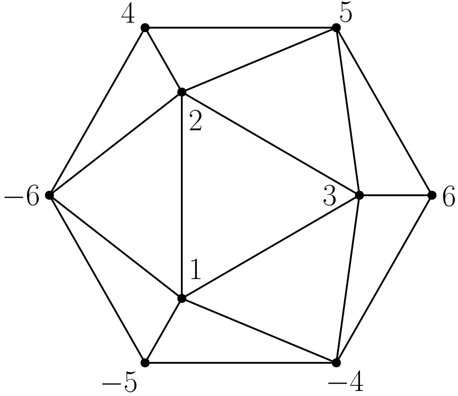
$\mathbb{R}P^1$ example



$\mathbb{R}P^1$ non-example



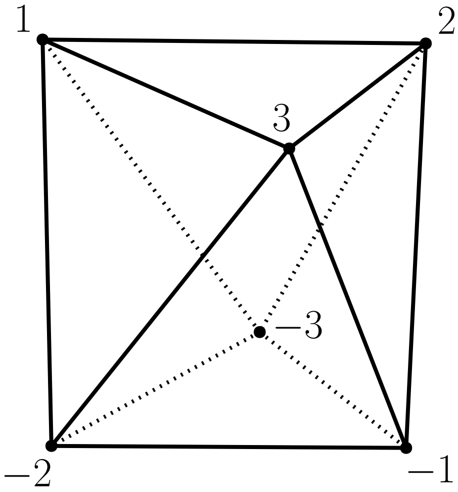
$\mathbb{R}P^2$ example



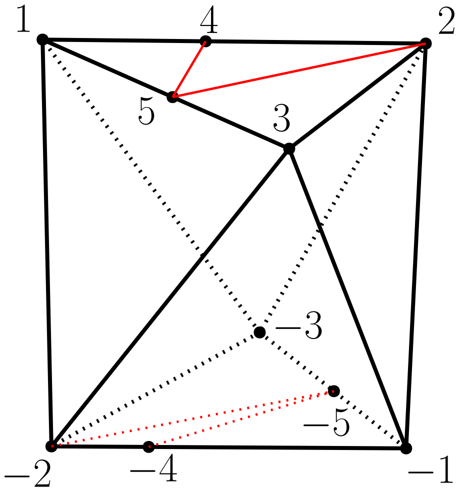
Our construction

- Start with a crosspolytope with vertices $\{\pm 1, \pm 2, \ldots, \pm n\}$. The facet $\{1, 2, \ldots, n\}$ is called *positive* and $\{-1, -2, \ldots, -n\}$ is called *negative*.
- Triangulate the positive facet with a certain triangulation T adding additional vertices. Triangulate the negative facet symmetrically.
- Each side facet is a join $\sigma * (-\tau)$ of faces σ and $-\tau$ belonging to the positive and the negative facet, resp. Moreover σ and τ are such that $\sigma \cap \tau = \emptyset$.
  We have already triangulated both σ and $-\tau$, so triangulate $\sigma * (-\tau)$ as their join without adding new vertices.

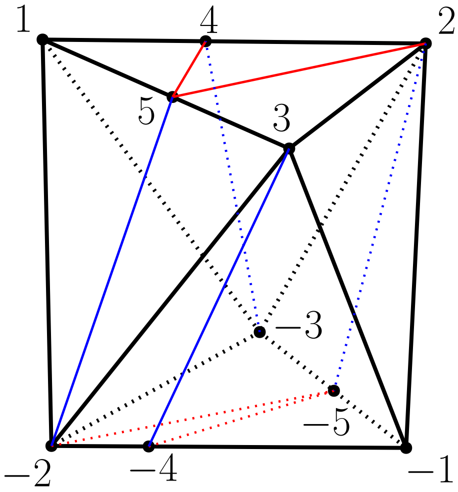
Our construction



Our construction



Our construction

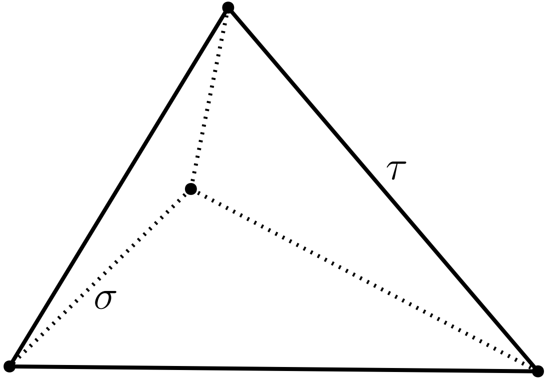


Triangulation of a join

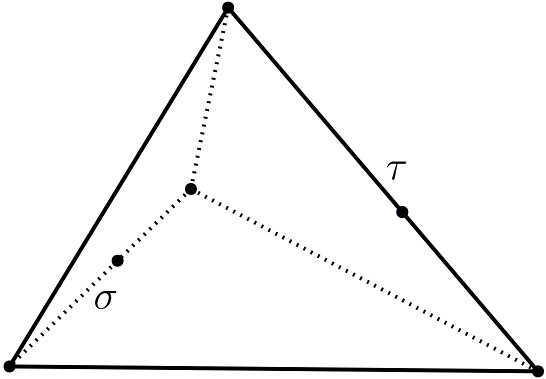
Suppose we are given triangulated simplices (or simplicial complexes in general) σ and τ.

The join $\sigma * \tau$ is the triangulated in the following way: for every pair of simplices $\sigma' \subset \sigma$ and $\tau' \subset \tau$ the triangulation of $\sigma * \tau$ contains a simplex $\sigma' * \tau'$.

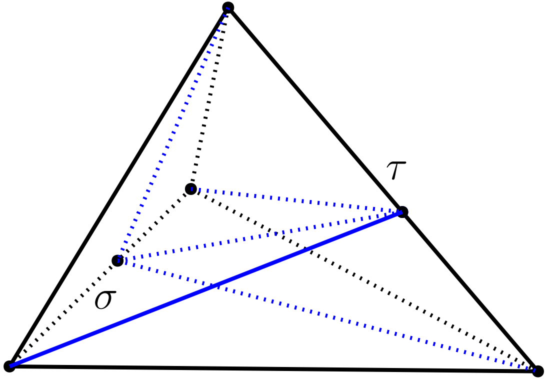
Triangulation of a join



Triangulation of a join



Triangulation of a join



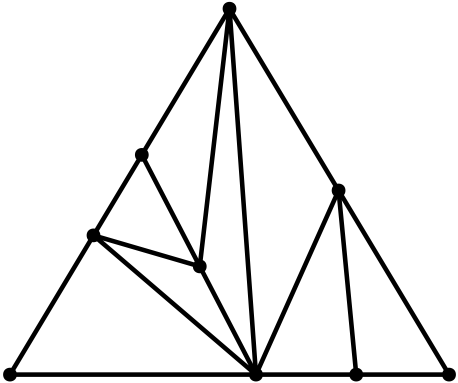
What do we need from T?

Suppose that in our refining of the crosspolytope triangulation there is a path $x \to y \to -x$ between x and $-x$. Wlog both x and y are in the positive facet. Then the edge $y \to -x$ is in a side facet $\sigma * (-\tau)$ where $\sigma \cap \tau = \emptyset$.

So, triangulation T of the positive facet is *good* if for any two faces σ, τ with $\sigma \cap \tau = \emptyset$ there is no edge between σ and τ.

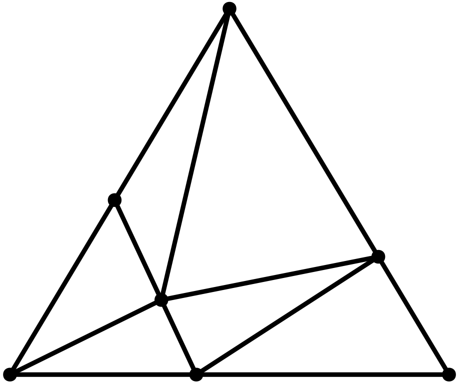
What do we need from T?

Bad triangulation



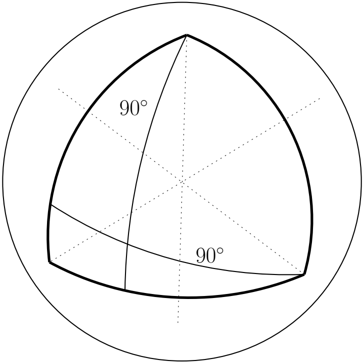
What do we need from T?

Good triangulation



Spherical interpretation

In the spherical simplex, for any $\sigma \cap \tau = \emptyset$ any edge between σ and τ has length 90°.

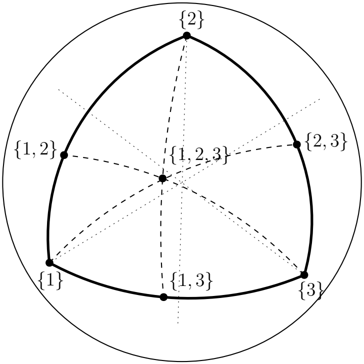


So, T is good if all the edges are shorter than 90°.

Vertex notation

The vertices of T will be *some* of the vertices of the barycentric subdivision.

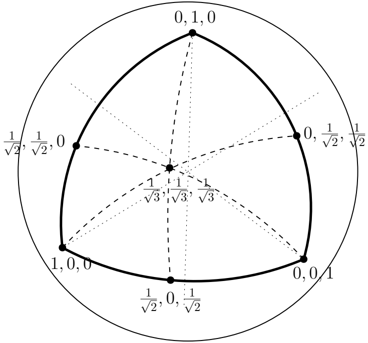
Vertices are identified with the subsets of $\{1, 2, \ldots, n\}$ ...



Vertex notation

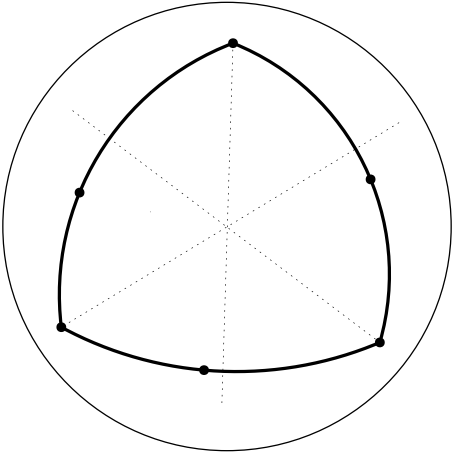
The vertices of T will be *some* of the vertices of the barycentric subdivision.

... and with unit vectors.



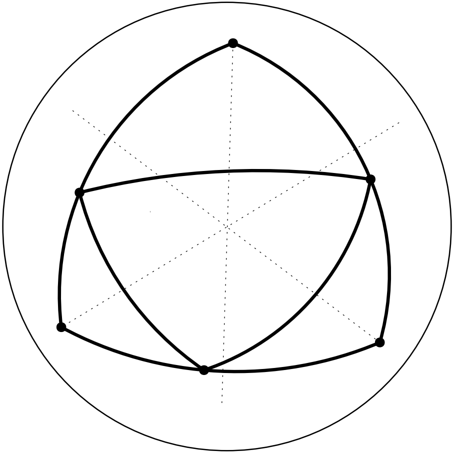
Construction of T

Take a certain subset $V \subset 2^{|n|}$ - some of the vertices of the barycentric subdivision. Take T to be the *Delaunay triangulation* on V.



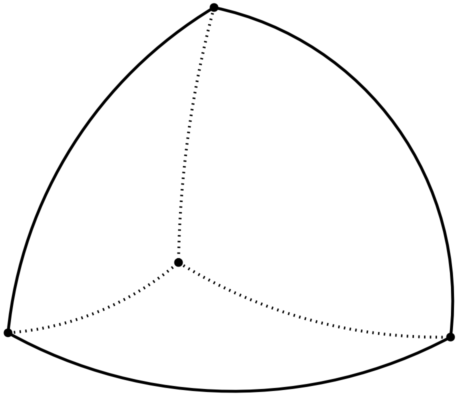
Construction of T

Take a certain subset $V \subset 2^{|n|}$ - some of the vertices of the barycentric subdivision. Take T to be the *Delaunay triangulation* on V.



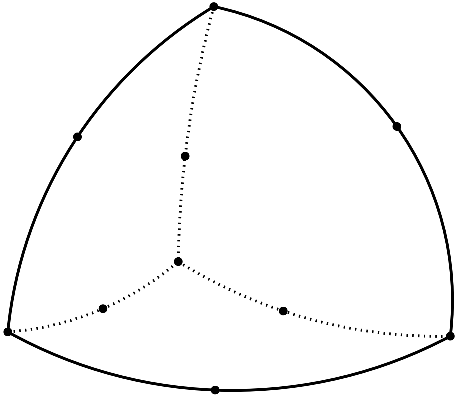
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



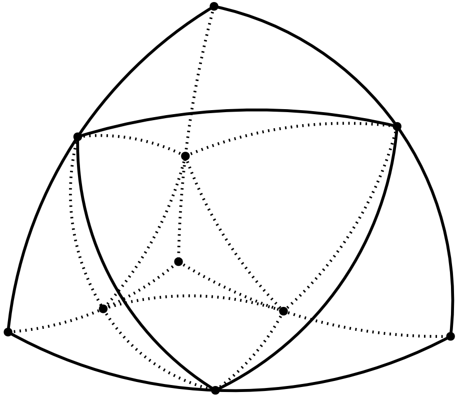
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



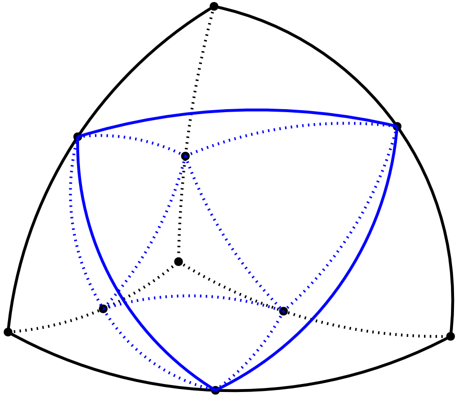
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



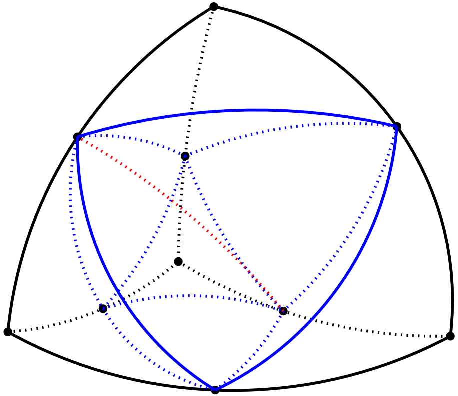
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



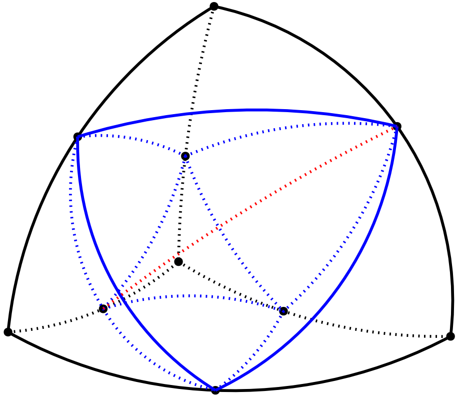
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



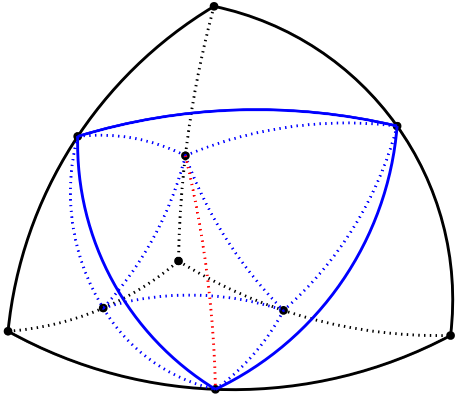
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



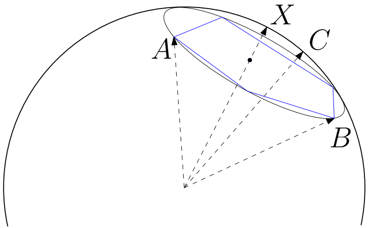
What do we need from V?

Taking as V all the vertices and all the edges midpoints is not enough. Consider the tetrahedron



What do we need from V?

The edges of T are shorter than 90° if for any $A, B \in V$ such that $\langle A, B \rangle = 0$ and any $X \in S^{n-1}$ there is $C \in V$ such that $\langle X, C \rangle > \langle X, A \rangle$ or $\langle X, C \rangle > \langle X, B \rangle$.



We can assume that the coordinates of X are *non-negative*.

Sufficient conditions on V

- $\{i\} \in V$ for all $i \in \{1, \ldots, n\}$.
- If $A \in V$ and $|A| > 1$, then $A \setminus i \in V$ for any $i \in A$.
- For every $A, B \in V$ with $A \cap B = \emptyset$, there are $i \in A$ and $j \in B$ such that either
  - $A \sqcup j \in V$ and $B \sqcup i \setminus j \in V$,

  or
  - $B \sqcup i \in V$ and $A \sqcup j \setminus i \in V$.

Proof of sufficiency

Wlog

$$
\begin{array}{lcccccccc}
A = & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & 0 & \dots & 0 \\
B = & 0 & \dots & 0 & \dots & 0 & * & \dots & \frac{1}{\sqrt{b}} \\
X = & x_1 \leq & \dots & \leq x_i \leq & \dots & \leq x_a & x_{a+1} & \dots & x_j
\end{array}
$$

Proof of sufficiency

Wlog

$$\begin{array}{lcccccccc}
A = & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & 0 & \dots & 0 \\
B = & 0 & \dots & 0 & \dots & 0 & * & \dots & \frac{1}{\sqrt{b}} \\
X = & x_1 \leq & \dots & \leq x_i \leq & \dots & \leq x_a & x_{a+1} & \dots & x_j
\end{array}$$

Assume $x_j < x_i$. Then $C := B \sqcup i \setminus j \in V$ is closer to X than B is. So,

$$x_j \geq x_i \geq x_1.$$

Proof of sufficiency

Consider $C_1 := A \setminus 1 \in V$ and $C_2 := A \sqcup j \in V$.

$$
\begin{array}{lccccccccc}
A = & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & \dots & \frac{1}{\sqrt{a}} & 0 & \dots & 0 & 0 \\
C_1 = & \boxed{0} & \dots & \frac{1}{\sqrt{a-1}} & \dots & \frac{1}{\sqrt{a-1}} & 0 & \dots & 0 & 0 \\
C_2 = & \frac{1}{\sqrt{a+1}} & \dots & \frac{1}{\sqrt{a+1}} & \dots & \frac{1}{\sqrt{a+1}} & 0 & \dots & 0 & \boxed{\frac{1}{\sqrt{a+1}}} \\
X = & x_1 \leq & \dots & \leq x_i \leq & \dots & \leq x_a & x_{a+1} & \dots & x_{j-1} & x_j
\end{array}
$$

We have

$$\langle X, A \rangle = \frac{x_1 + \cdots + x_a}{\sqrt{a}},$$

$$\langle X, C_1 \rangle = \frac{x_2 + \cdots + x_a}{\sqrt{a-1}},$$

$$\langle X, C_2 \rangle = \frac{(x_1 + x_j) + x_2 + \cdots + x_a}{\sqrt{a+1}} \geq \frac{2x_1 + x_2 + \cdots + x_a}{\sqrt{a+1}}.$$

Proof of sufficiency

$$\langle X, A\rangle = \frac{x_1 + \cdots + x_a}{\sqrt{a}}, \langle X, C_1\rangle = \frac{x_2 + \cdots + x_a}{\sqrt{a-1}},$$

$$\langle X, C_2\rangle \geq \frac{2x_1 + x_2 + \cdots + x_a}{\sqrt{a+1}}.$$

Denote

$$f(\alpha) := \frac{\alpha x_1 + x_2 + \cdots + x_a}{\sqrt{a-1+\alpha}} =$$

$$= x_1\sqrt{a-1+\alpha} + \frac{(x_2 - x_1) + \cdots + (x_a - x_1)}{\sqrt{a-1+\alpha}}.$$

We have that

$$\langle X, A\rangle = f(1), \quad \langle X, C_1\rangle = f(0), \quad \langle X, C_2\rangle \geq f(2).$$

Proof of sufficiency, end

$$\langle X, A\rangle = f(1), \quad \langle X, C_1\rangle = f(0), \quad \langle X, C_2\rangle \geq f(2).$$

$$f(\alpha) = x_1\sqrt{a-1+\alpha} + \frac{(x_2 - x_1) + \cdots + (x_a - x_1)}{\sqrt{a-1+\alpha}}.$$

From $0 \leq x_1 \leq x_2 \cdots \leq x_a$ we have that f as a function of $\sqrt{a-1+\alpha}$ is either

- convex,
- or linear non-constant,
- or zero.

In the first two cases either $\langle X, C_1\rangle \geq \langle X, A\rangle$ or $\langle X, C_2\rangle \geq \langle X, A\rangle$. In the last case, $\langle X, A\rangle = 0$ and there is a singleton $C \in V$ such that $\langle X, C\rangle > 0$.

Constructing a small set V

Partition the set $\{1, \dots, n\}$ into several disjoint *groups*.

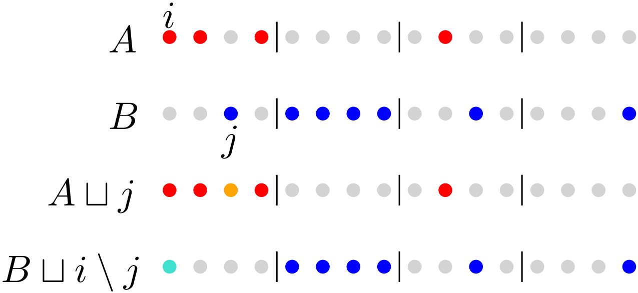
Let V be the set of subsets of $\{1, \dots, n\}$, whose intersection with every group, except maybe one, contains not more than one element.

Clearly,

- $\{i\} \in V$ for all $i \in \{1, \dots, n\}$.
- If $A \in V$ and $|A| > 1$, then $A \setminus i \in V$ for any $i \in A$.

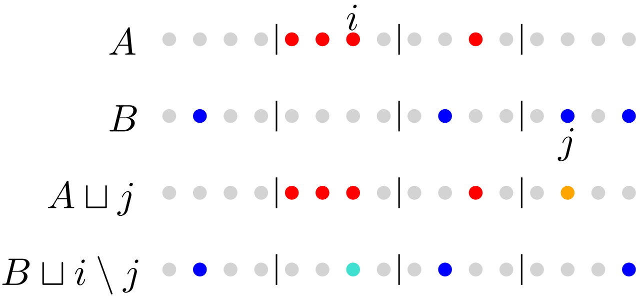
Constructing a small set V

Let us check the last required property of V.
Case 1:



Constructing a small set V

Let us check the last required property of V.
Case 2:



Size of V

We use k groups of size roughly $s = n/k$. For an element $A \in V$ we have

- k choices of the maximal group of A,
- at most 2^s choices of which elements of the maximal group to add to A,
- at most $s+1$ choices for each of the $k-1$ elements of A in the non-maximal groups.

In total we get

$$|V| < 2^s (s+1)^{k-1} k.$$

Size of V

$$|V| < 2^s(s+1)^{k-1}k.$$

Choosing $k = s = \sqrt{n}$ we get

$$|V| < e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}.$$

Lower bound

Theorem (Arnoux, Marin)

Let P be a simplicial complex with a 1-cocycle ω such that $\omega^n \neq 0$. Then P has at least $\frac{(n+2)(n+1)}{2}$ vertices.

Corollary

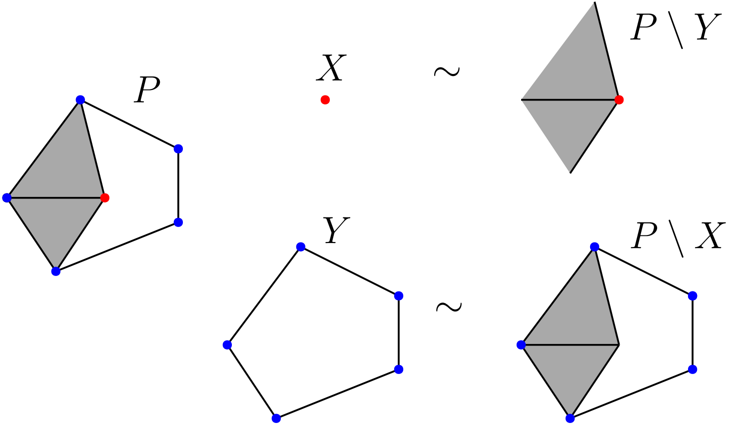
$|\mathbb{R}P^n| \geq \frac{(n+2)(n+1)}{2}$.

Proof of the lower bound

Suppose that the set of vertices of P is decomposed into two disjoint sets. Denote by X and Y the subcomplexes induced by the sets. We have that

$$X \sim P \setminus Y,$$
$$Y \sim P \setminus X,$$
$$P = (P \setminus X) \cup (P \setminus Y).$$

Proof of the lower bound



Proof of the lower bound, end

Complex P has a non-trivial n-cocycle ω^n. So, there is at least one n-simplex $\Delta^n \subset P$.

Take $Y = \Delta^n$. Then $\omega|_{P \setminus X} = \omega|_Y = 0$.

Suppose that $(\omega|_X)^{n-1} = (\omega|_{P \setminus Y})^{n-1} = 0$. Then from $P = (P \setminus X) \cup (P \setminus Y)$ we get that $\omega \cdot \omega^{n-1} = \omega^n = 0$, contradiction.

So, $(\omega|_X)^{n-1} \neq 0$. By induction, X has at least $\frac{(n+1)n}{2}$ vertices. So, P has at least $\frac{(n+1)n}{2} + (n+1) = \frac{(n+2)(n+1)}{2}$ vertices.

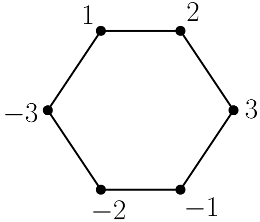
Facets lower bound

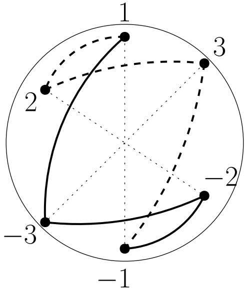
Theorem (Bárány, Lovász)

Any symmetric triangulation of S^n has at least 2^{n+1} facets.

Proof of the facets lower bound

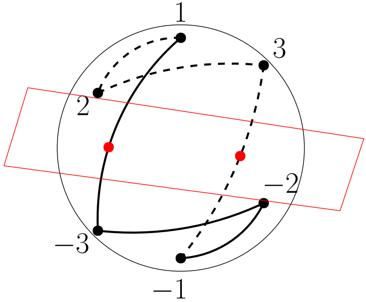
Embed the given triangulated S^n into the unit sphere $S^{|V/2|-1}$, where V is the number of vertices in the triangulation.





Proof of the facets lower bound, end

By the Borsuk–Ulam theorem, any central hyperplane of codimension n intersects at least two facets.



By the Crofton formula, the total n-dimensional volume of the embedded S^n is at least the same as the volume of the unit n-sphere.

The n-volume of each facet is $\frac{1}{2^{n+1}}$. So, the total number of facets is at least 2^{n+1}.