Exact semidefinite programming bounds for packing problems

Philippe Moustrou, UiT - The Arctic University of Norway
Joint work with M. Dostert (KTH) and D. de Laat (TU Delft).

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How can we turn these bounds into exact bounds?

Motivation: the kissing number problem

How many unit spheres can simultaneously touch a central unit sphere without overlapping?

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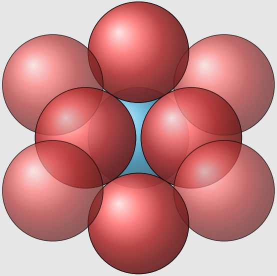

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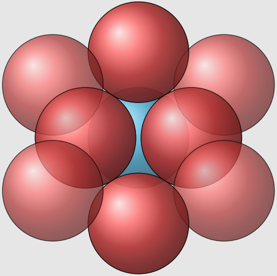

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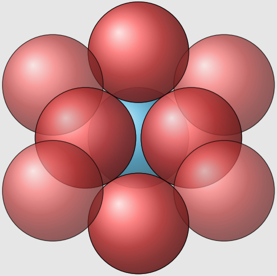
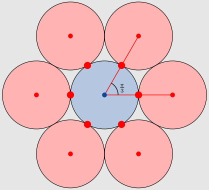


Known in dimensions 1, 2, 3 (Schutte, vander Waerden, 1953),

4 (Musin, 2008), 8 and 24 (Levenshtein / Odlyzko, Sloane, 1979).

Formulation and generalizations



Kissing number:

$$\max\{|C|, \quad C \subset S^{n-1}, \quad x \cdot y \leq 1/2 \text{ for all } x \neq y \in C\}$$

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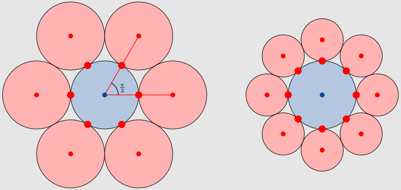

Spherical codes:

$$\max\{|C|, \quad C \subset S^{n-1}, \quad x \cdot y \leq \cos\theta \text{ for all } x \neq y \in C\}$$

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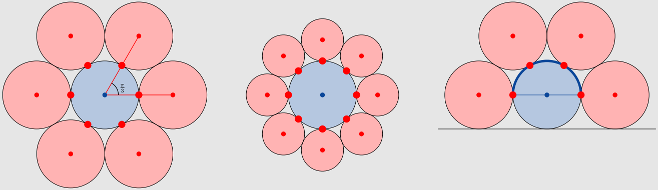

One-sided kissing number (Musin, 2006):

$$\max\{|C|, \quad C \subset \mathrm{H}^{n-1}, \quad x \cdot y \leq 1/2 \text{ for all } x \neq y \in C\}$$

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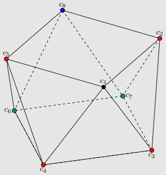
- The square antiprism, the unique optimal θ-spherical code in dimension 3 with $\cos\theta = (2\sqrt{2}-1)/7$ (Schütte-van der Waerden 1951, Danzer 1986).



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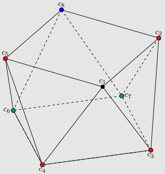
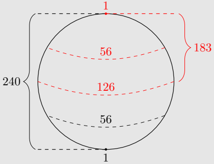


- For the Hemisphere in dimension 8: the E_8 lattice provides an optimal configuration (Bachoc-Vallentin, 2008). What about uniqueness?



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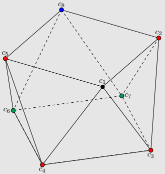

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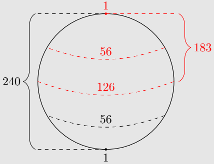

- [Dostert, De Laat, M., 2020]: A general framework to prove optimality and uniqueness of such configurations.

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- Upper bounds:
  - For finite graphs: hierarchies of semidefinite upper bounds. (Lovász-Schrijver 1991, Lasserre 2001, Laurent 2007)
  - For infinite graphs: Generalization of Lasserre's hierarchy (de Laat-Vallentin 2015), related to the previous 2-point (Delsarte-Goethals-Seidel 1977) and 3-point bounds (Bachoc-Vallentin 2008).

2-point bound for spherical codes (Delsarte-Goethals-Seidel 1977)

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- Up to symmetry, a couple x, y of points in a θ-spherical code is uniquely determined by

$$u = x \cdot y, \quad \text{with } \begin{cases} u = 1 & x = y \\ u \in [-1, \cos\theta] & x \neq y \end{cases}$$

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- The normalized Gegenbauer polynomials $P_k^n(u)$ (with $P_k^n(1) = 1$), satisfying:

$$\text{For every } X \subset S^{n-1} \text{ finite, } \sum_{x,y \in X} P_k^n(x \cdot y) \geq 0.$$

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Assume we have a polynomial f such that

- there exists coefficients $\alpha_0, \dots, \alpha_d \geq 0$ such that

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So

$$|C| \leq f(1) + 1$$

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So for every $d \geq 0$, the size of a θ-spherical code is at most

$$\begin{aligned}\min\{M \in \mathbb{R} : \alpha_0, \ldots, \alpha_d &\geq 0,\\ f(1) &\leq M - 1,\\ f(u) &\leq -1 \text{ for all } u \in [-1, \cos\theta]\}\end{aligned}$$

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This is a linear programming bound.

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with (u, v, t) in

$$\begin{cases} \{(1,1,1)\} & x = y = z \\ \Delta_0 = \{(u,u,1) : u \in [-1, \cos\theta]\} & x \neq y = z \\ \Delta & x, y, z \text{ distinct} \end{cases}$$

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- Matrix polynomials $S_k^n(u, v, t)$ satisfying:

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Then for every $d \geq 0$, the size of a θ-spherical code is at most

$$\begin{aligned}\min\{M \in \mathbb{R} : \alpha_k \geq 0, F_k \succeq 0\\ &\sum_{k=0}^{d} \alpha_k + F(1,1,1) \leq M - 1,\\ &\sum_{k=0}^{d} \alpha_k P_k^n(u) + 3F(u,u,1) \leq -1 \text{ for all } u \in [-1, \cos\theta],\\ &F(u,v,t) \leq 0 \text{ for all } (u,v,t) \in \Delta\}\end{aligned}$$

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$$F(u,v,t) = \sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t)\rangle.$$

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$$F(u,v,t) = \sum_{k=0}^{d} \langle F_k, S_k^n(u,v,t) \rangle.$$

This leads to semidefinite upper bounds using sums of squares.

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So why do we want an exact sharp bound?

- Optimization: When does a bound give the independence number?
- Geometry: Sharp bounds provide additional information on optimal configurations, leading to uniqueness proofs.

Recap

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- For spherical codes, including kissing number:
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- For spherical codes in spherical caps, like hemisphere:
  - Delsarte bound does not apply anymore due to the lack of symmetry.
  - The 3-point bound can be adapted to a 2-point semidefinite programming bound (Bachoc-Vallentin 2009).

Results

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- Numerically sharp for the square antiprism (Bachoc-Vallentin 2009)
  $\rightarrow$ Rigorous proof (Dostert-de Laat-M 2020)
- E_8 gives an optimal configuration on the hemisphere in dimension 8 (Bachoc-Vallentin 2009)
  $\rightarrow$ Uniqueness (Dostert-de Laat-M 2020)

Solving an SDP: Rage against the machine precision

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A semidefinite program:

$$\inf\{\ \underbrace{c^t x}_{\text{objective}}\ :\ \underbrace{Ax = b}_{\text{linear constraints}}\ ,\ \underbrace{\mathcal{B}_i(x) \succeq 0}_{\text{PSD constraints}}\ \}$$

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- Our context: The problems provide a candidate field to round over, either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{d})$.

Rounding over $\mathbb{Q}$: the affine conditions

Solve the SDP numerically in high precision (SDPA-GMP),

$\rightarrow$ get an approximate solution x^*:

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The linear system is then satisfied... But what about the PSD conditions?

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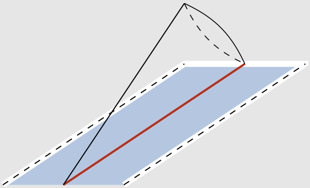

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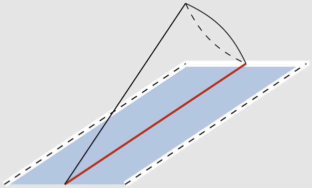

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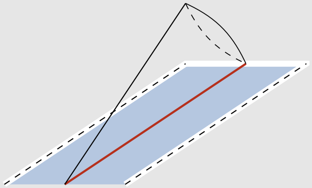

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- Sometimes not. By undertsanding the kernels, we can force all these constraints!

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Thank you!