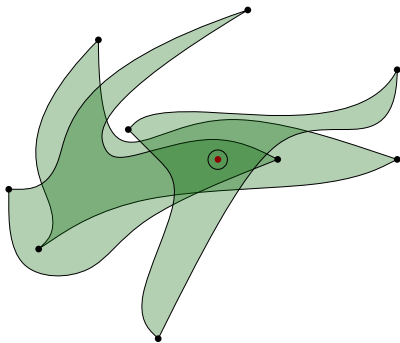


The topological Tverberg problem beyond prime powers.

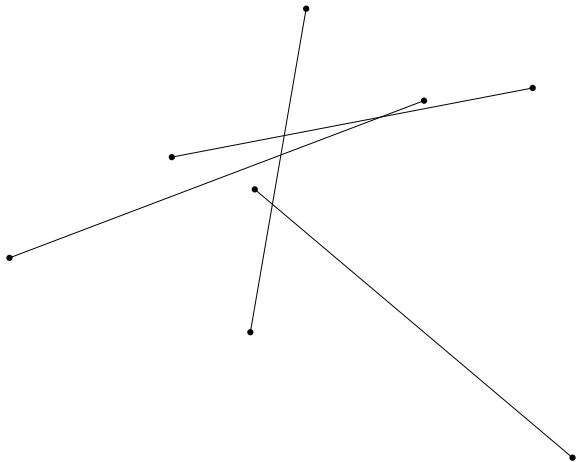


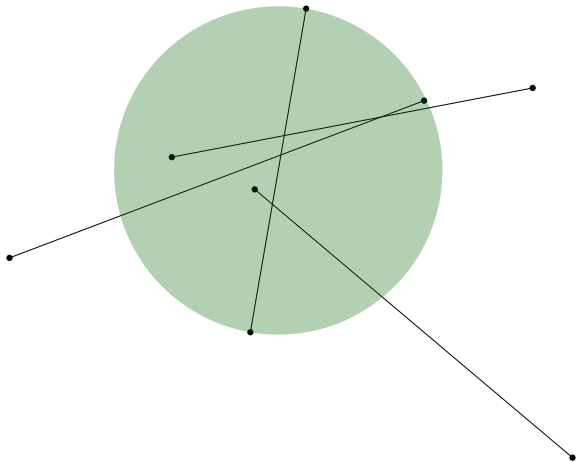
Pablo Soberón
Baruch College, City University of New York

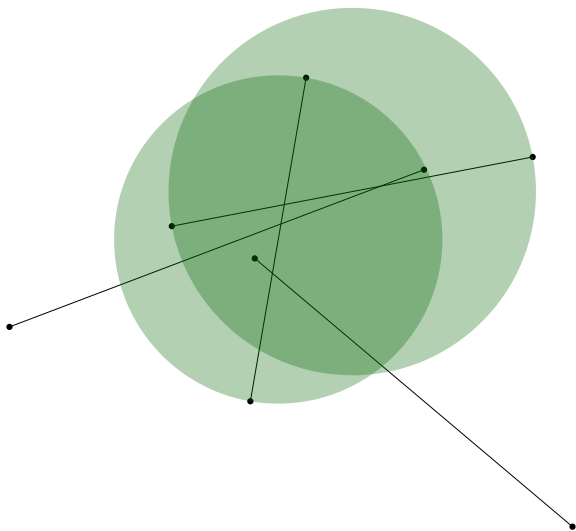
A warm-up

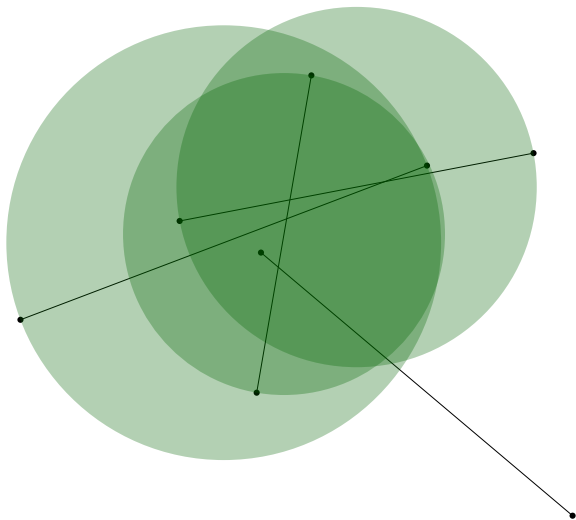
A warm-up

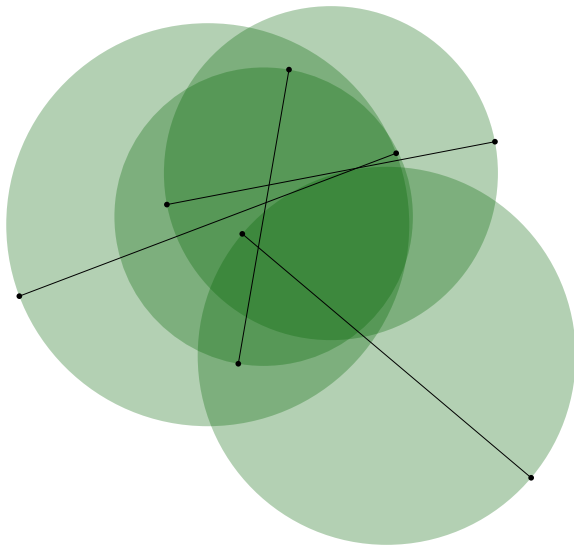


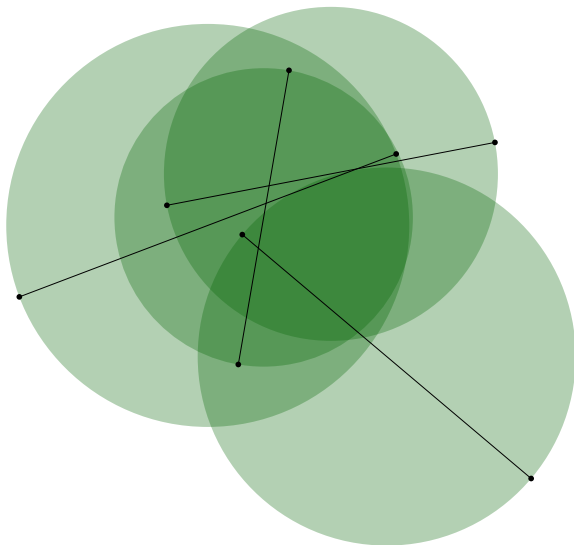






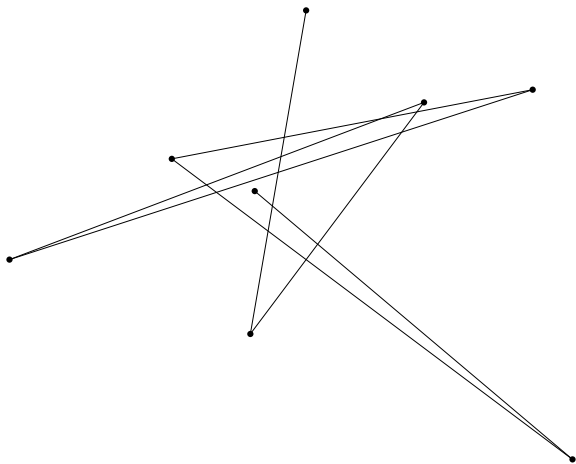






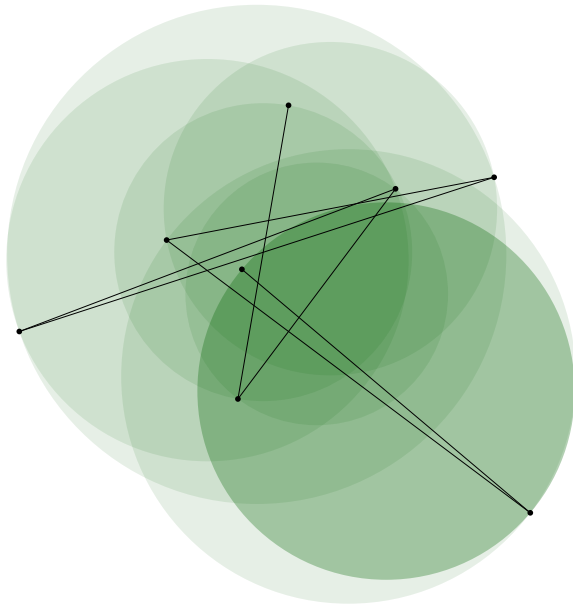
(Huemer, Pérez-Lantero, Seara, Silveira 2019)

For any $2n$ points on the plane, there exists a perfect matching whose induced disks intersect



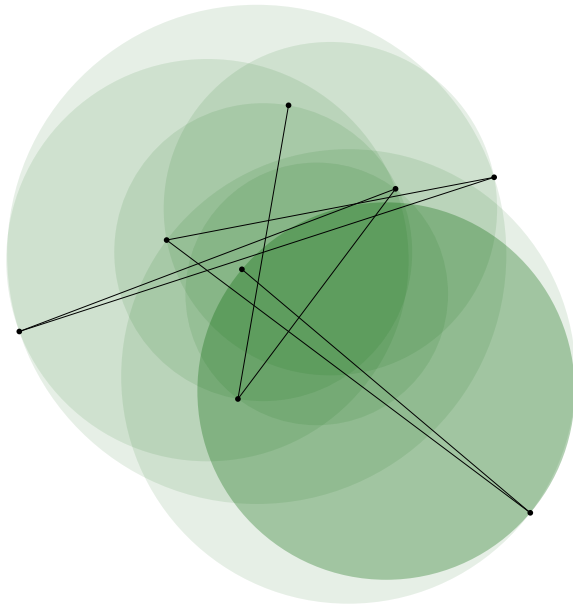
(Huemer, Pérez-Lantero, Seara, Silveira 2019)

For any $2n$ points on the plane, there exists a perfect matching whose induced disks intersect



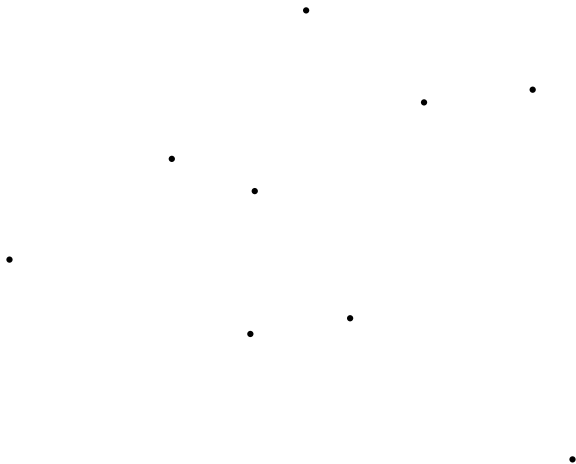
(Huemer, Pérez-Lantero, Seara, Silveira 2019)

For any $2n$ points on the plane, there exists a perfect matching whose induced disks intersect



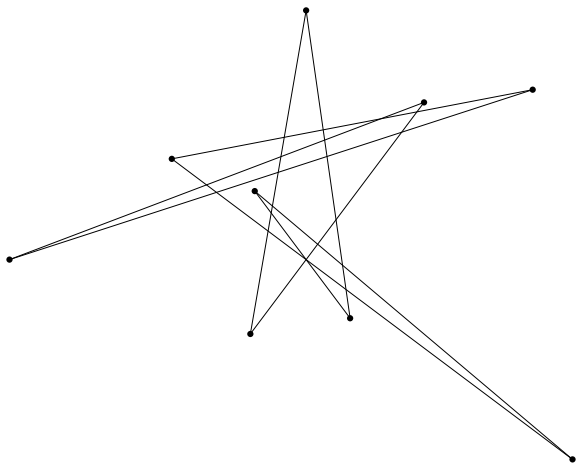
(Soberón, Tang 2020+)

For any $2n$ points on the plane, there exists a **Hamiltonian path** whose induced disks intersect



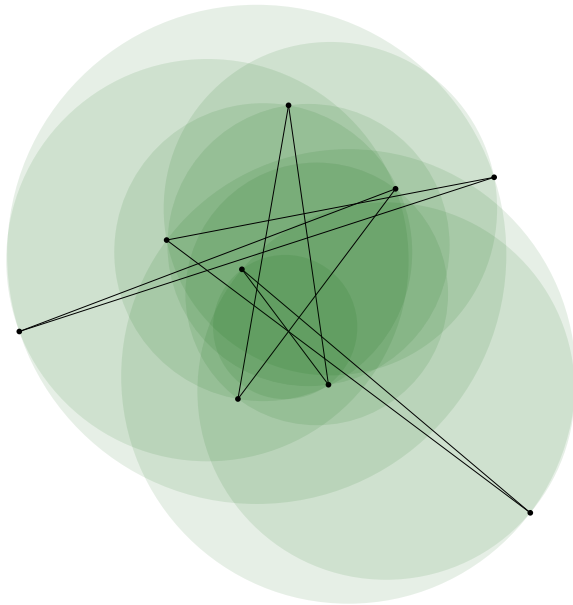
(Soberón, Tang 2020+)

For any $2n$ points on the plane, there exists a **Hamiltonian path** whose induced disks intersect



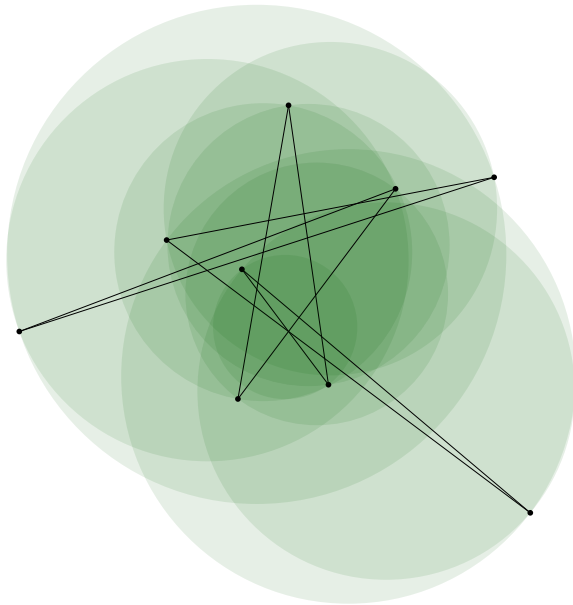
(Soberón, Tang 2020+)

For any $2n$ points on the plane, there exists a **Hamiltonian path** whose induced disks intersect



(Soberón, Tang 2020+)

For any $2n$ points on the plane, there exists a **Hamiltonian path** whose induced disks intersect



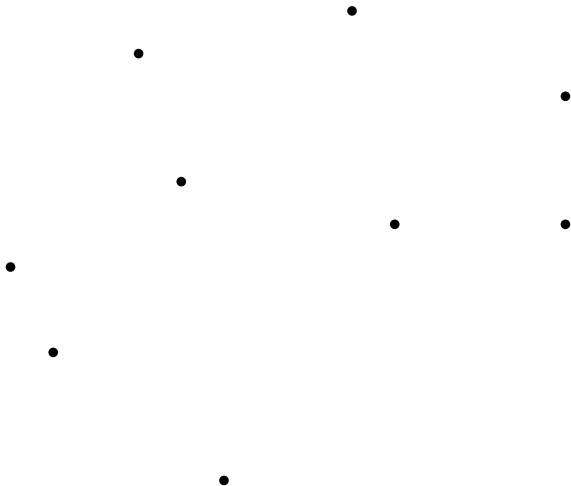
(Soberón, Tang 2020+)

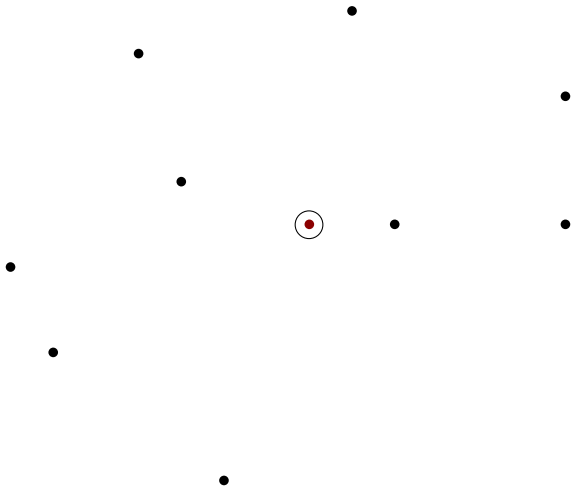
For any $2n + 1$ points on the plane, there exists a **Hamiltonian cycle** whose induced disks intersect

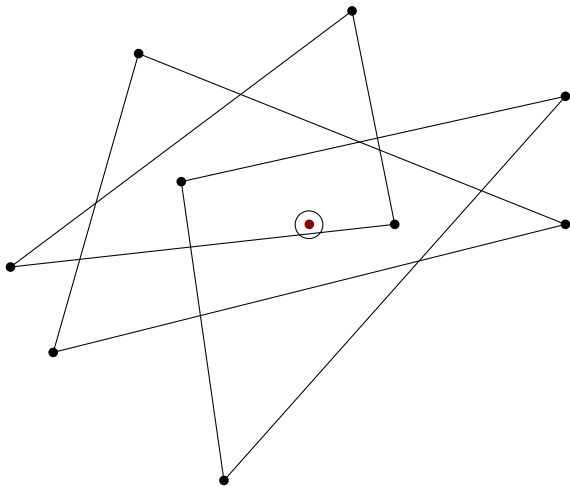
Now, the main topic.

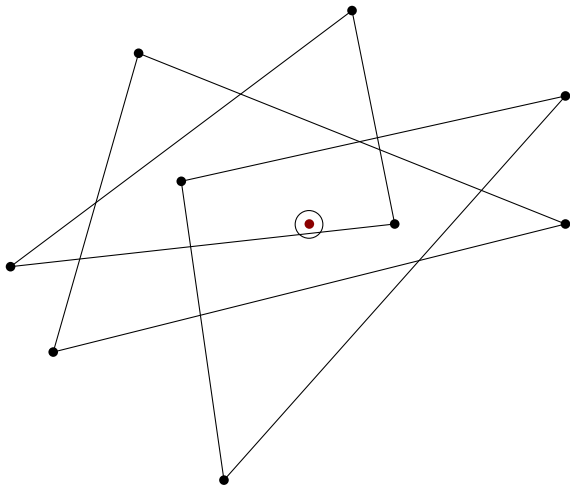
Joint work with Florian Frick.





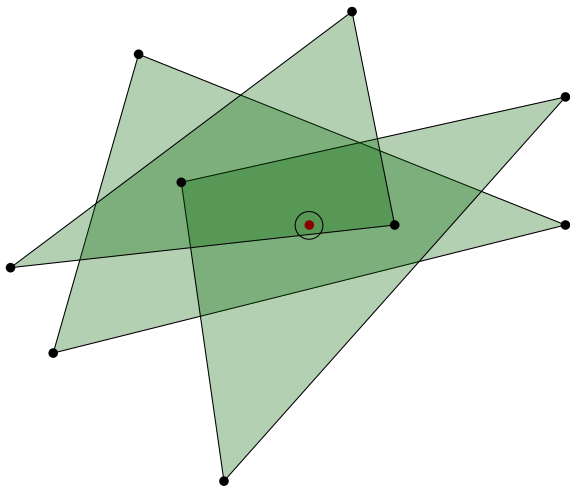






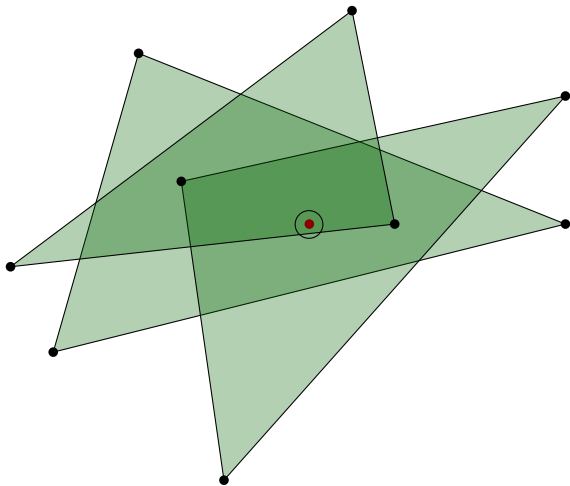
(Birch 1959)

Any $3q$ points in the plane can be split into q cycles that surround a common point.



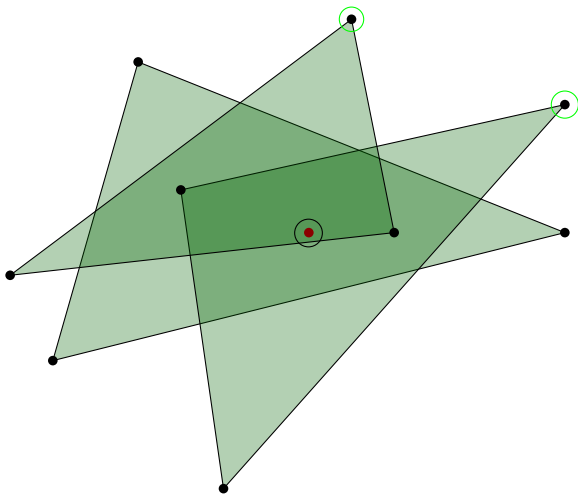
(Birch 1959)

Any $3q$ points in the plane can be split into q cycles that surround a common point.



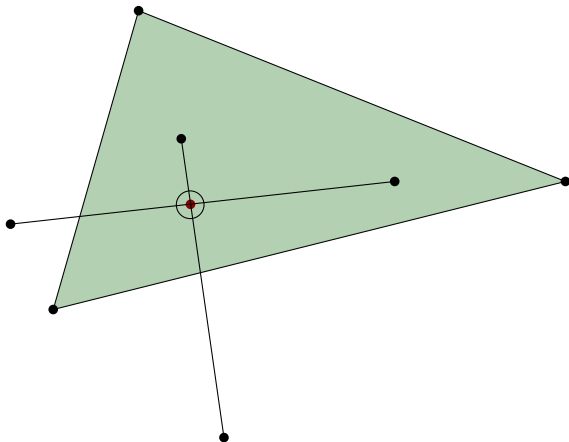
(Birch 1959)

Any $3q$ points in the plane can be split into q sets whose convex hulls intersect.



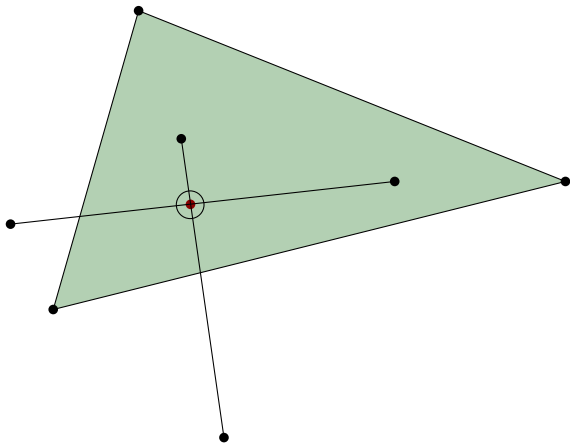
(Birch 1959)

Any $3q$ points in the plane can be split into q sets whose convex hulls intersect.



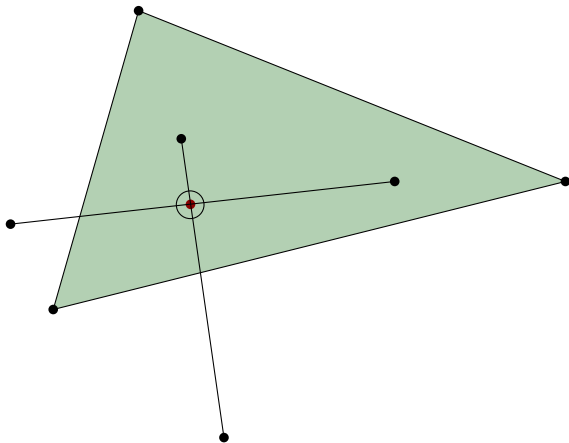
(Birch 1959)

Any $3q$ points in the plane can be split into q sets whose convex hulls intersect.



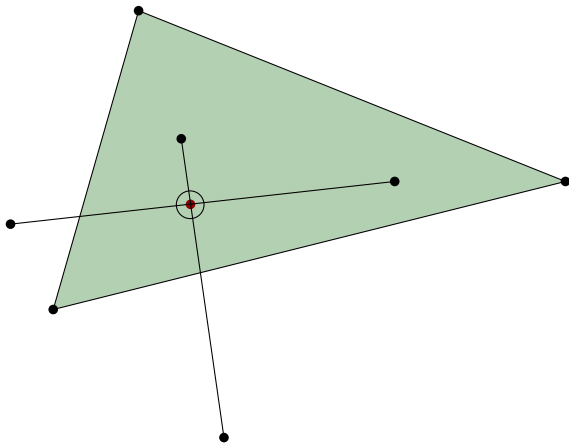
(Birch 1959)

Any $3q - 2$ points in the plane can be split into q sets whose convex hulls intersect.



(Tverberg 1966)

Any $(q-1)(d+1)+1$ points in \mathbb{R}^d can be split into q sets whose convex hulls intersect.



(Tverberg 1966)

Any $q(d+1) - d$ points in \mathbb{R}^d can be split into q sets whose convex hulls intersect.

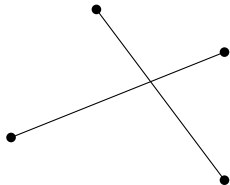
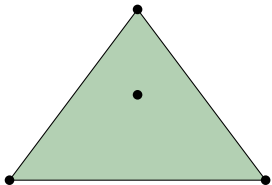
(Tverberg 1966)

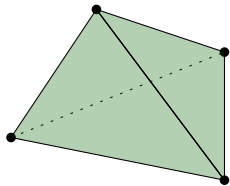
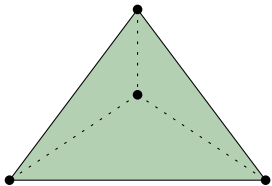
Any $q(d+1) - d$ points in \mathbb{R}^d can be split into q sets whose convex hulls intersect.

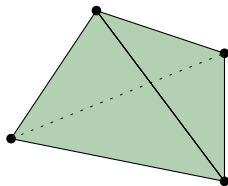
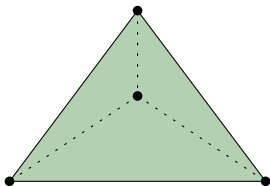
(Tverberg 1966)

Any $q(d+1) - d$ points in \mathbb{R}^d can be split into q sets whose convex hulls intersect.

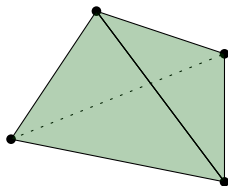
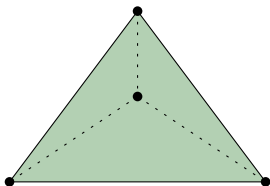






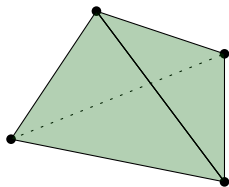
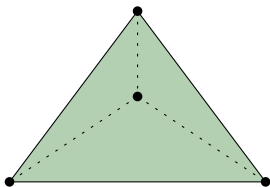


For any linear map $f : \Delta_3 \rightarrow \mathbb{R}^2$, there are 2 vertex-disjoint faces of Δ_3 whose images intersect.



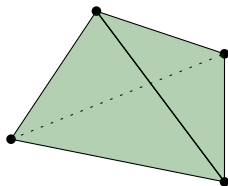
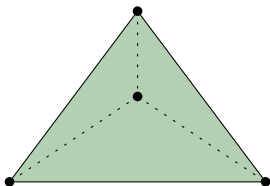
(Tverberg 1966)

For any linear map $f : \Delta_{(q-1)(d+1)} \rightarrow \mathbb{R}^d$, there are q vertex-disjoint faces of $\Delta_{(q-1)(d+1)}$ whose images intersect.



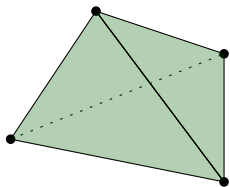
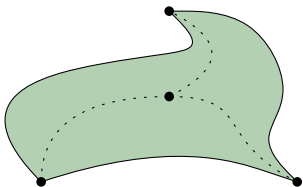
(Tverberg 1966)

For any **linear** map $f : \Delta_{(q-1)(d+1)} \rightarrow \mathbb{R}^d$, there are q vertex-disjoint faces of $\Delta_{(q-1)(d+1)}$ whose images intersect.



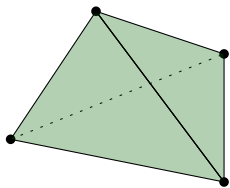
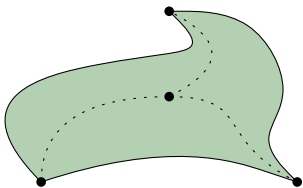
(Bárány, 1976) - Conjecture.

For any **continuous** map $f : \Delta_{(q-1)(d+1)} \rightarrow \mathbb{R}^d$, are there q vertex-disjoint faces of $\Delta_{(q-1)(d+1)}$ whose images intersect?



(Bárány, 1976) - Conjecture.

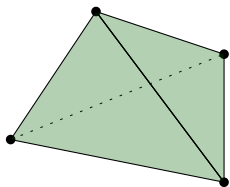
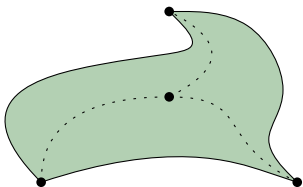
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(Barjmczy, Bárány, 1979) Yes! - For $q = 2$

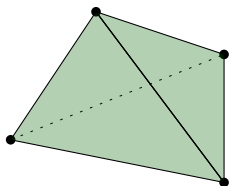
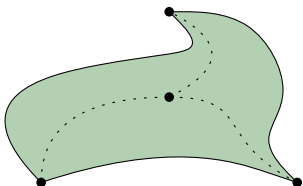


(Bárány, 1976) - Conjecture.

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(Barjóczy, Bárány, 1979) Yes! - For $q = 2$

(Bárány, Shlosman, Szűcs, 1981) Yes! - For q prime



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(Barjóczy, Bárány, 1979) Yes! - For $q = 2$

(Bárány, Shlosman, Szűcs, 1981) Yes! - For q prime

(Özaydin, 1987) Yes! - For q a prime power

It's _____'s fault!

It's Topology's fault!

It's Topology's fault!

(Frick, 2015) No! - For q not a prime power

It's Florian Frick's fault!

(Frick, 2015) No! - For q not a prime power

It's Isaac Mabillard's fault!

(Frick, 2015) No! - For q not a prime power

It's Uli Wagner's fault!

(Frick, 2015) No! - For q not a prime power

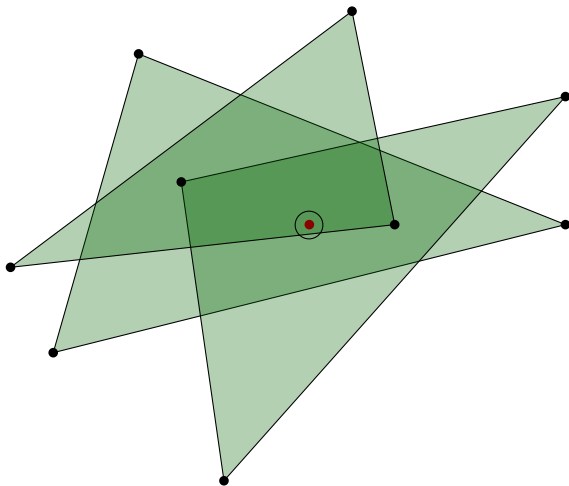
It's life's fault!

(Frick, 2015) No! - For q not a prime power

It's life's fault!

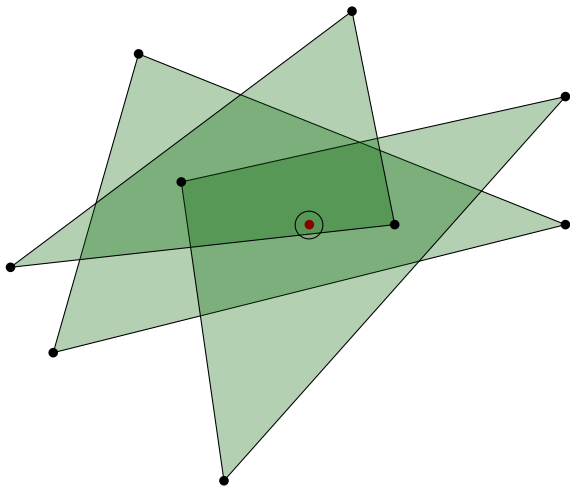
(Bárány, 1976) - Conjecture.

For any **continuous** map $f : \Delta_{(q-1)(d+1)} \rightarrow \mathbb{R}^d$, are there q vertex-disjoint faces of $\Delta_{(q-1)(d+1)}$ whose images intersect?



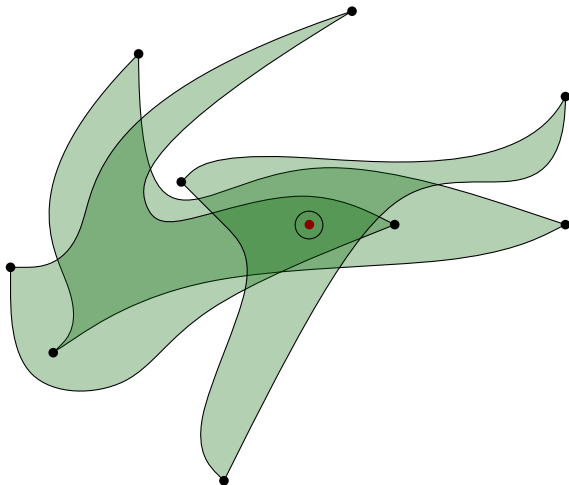
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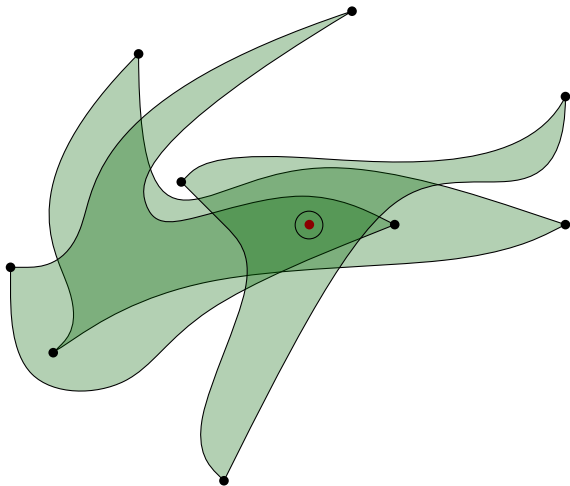
(Blagojević, Frick, Ziegler, 2014) - Conjecture.

For any **continuous** map $f : \Delta_{q(d+1)-1} \rightarrow \mathbb{R}^d$, are there q vertex-disjoint faces of $\Delta_{q(d+1)-1}$ whose images intersect?



(Blagojević, Frick, Ziegler, 2014) - Conjecture.

For any **continuous** map $f : \Delta_{q(d+1)-1} \rightarrow \mathbb{R}^d$, are there q vertex-disjoint faces of $\Delta_{q(d+1)-1}$ whose images intersect?



(Frick, Soberón, 2020+)

For any **continuous** map $f : \Delta_{q(d+1)-1} \rightarrow \mathbb{R}^d$, there are q vertex-disjoint faces of $\Delta_{q(d+1)-1}$ whose images intersect.

The German trick

The German trick

Add one point

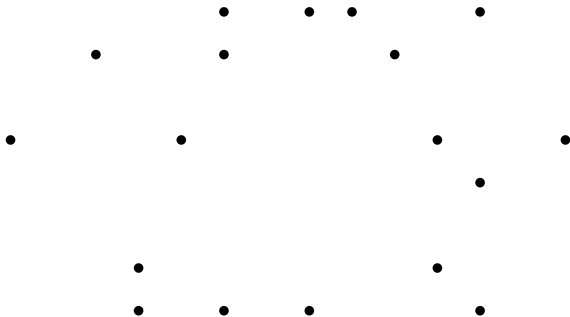
The German trick

Add one point
and ignore it.

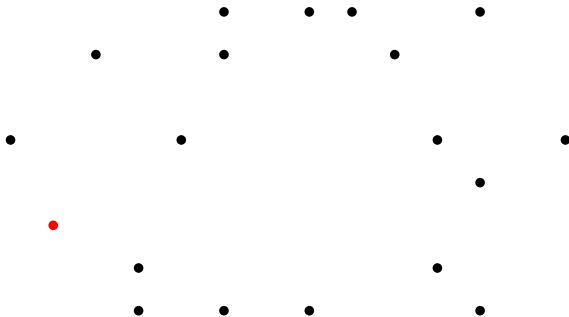
Suppose $q + 1$ is a prime power

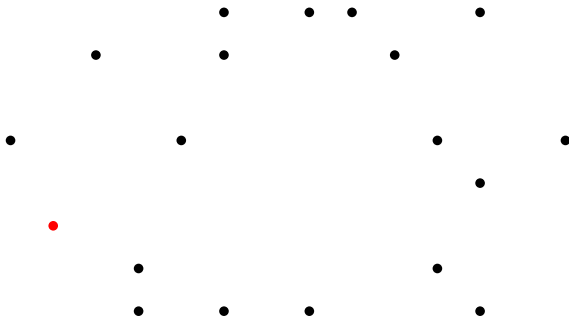
Suppose $q + 1$ is a prime power
and we have $q(d + 1)$ points

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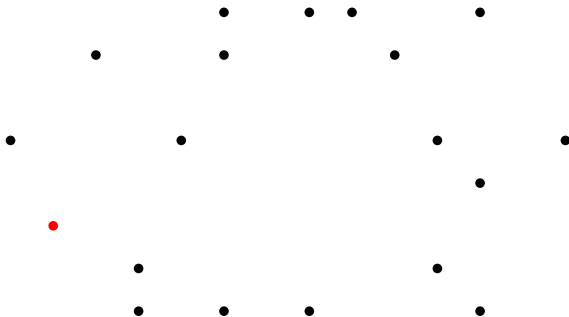


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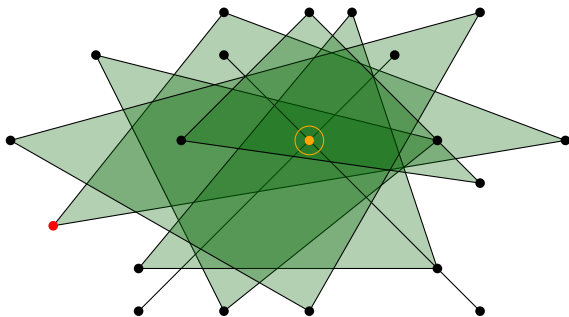




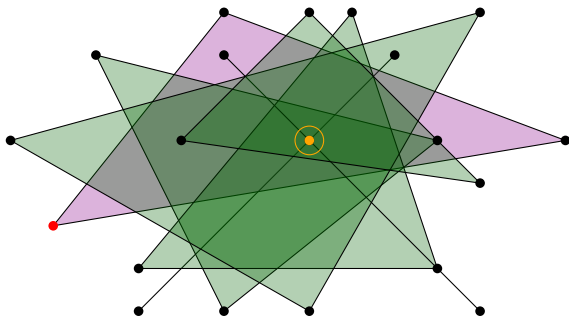
$$q(d+1) \rightarrow q(d+1) + 1$$



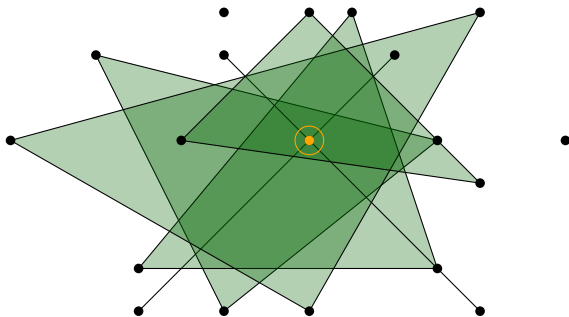
$$q(d+1) \rightarrow ((q+1)-1)(d+1)+1$$



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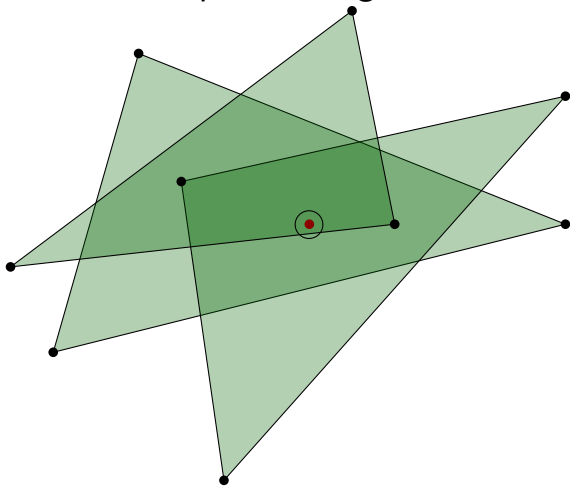
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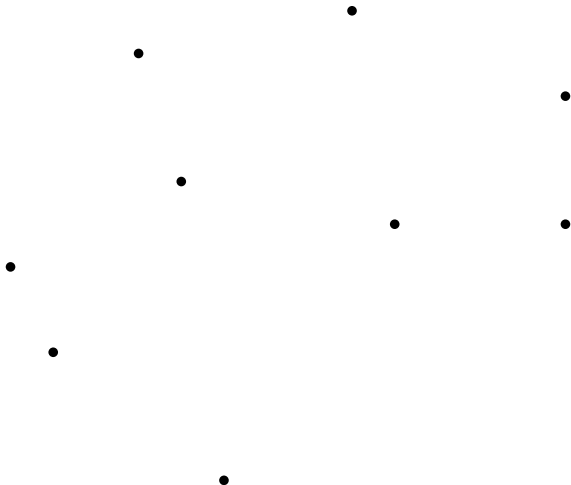


$$q(d+1) \rightarrow ((q+1)-1)(d+1)+1$$

How do we prove the general case?

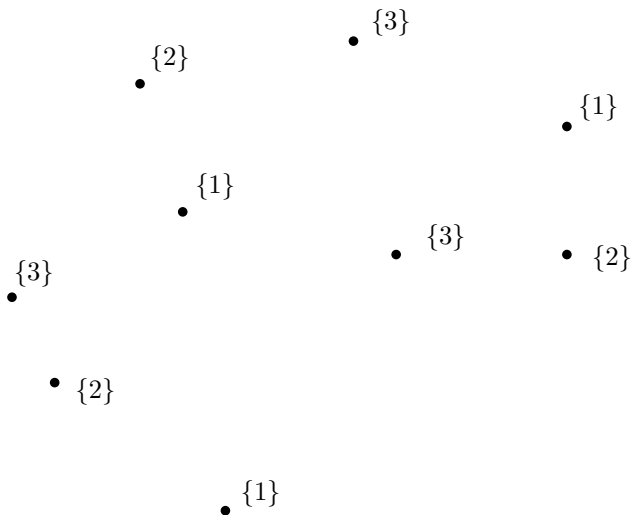
How do we prove the general case?



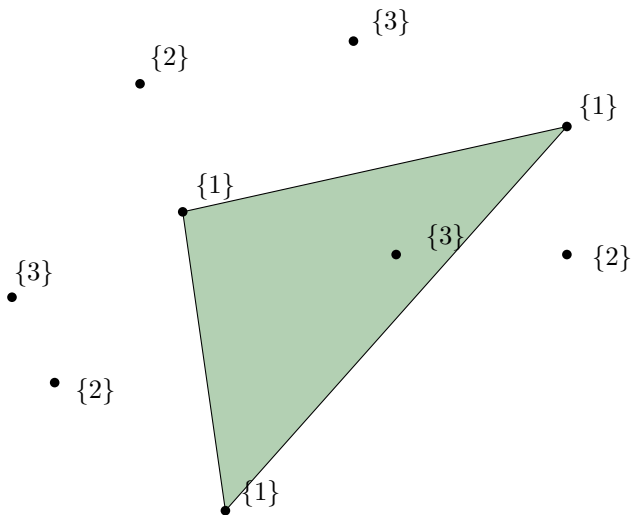




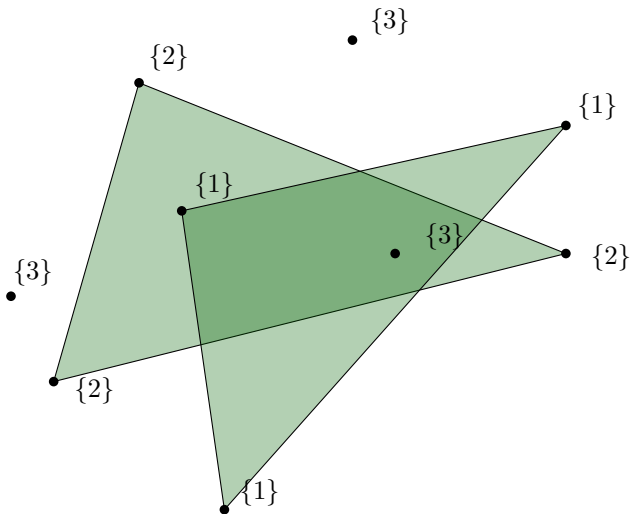
Assign to each vertex a label in $[q]$.



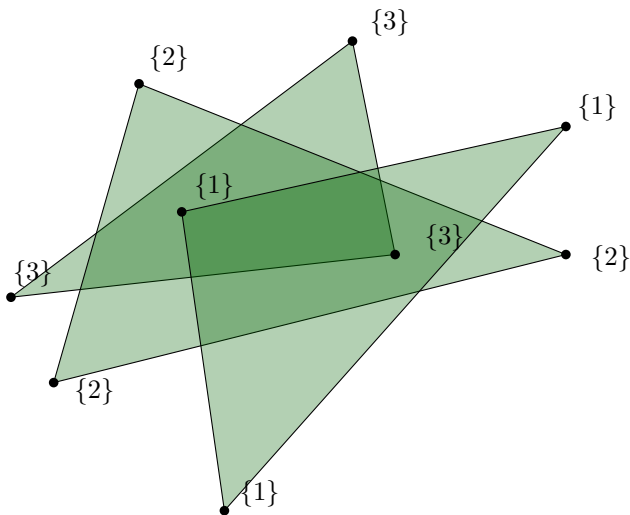
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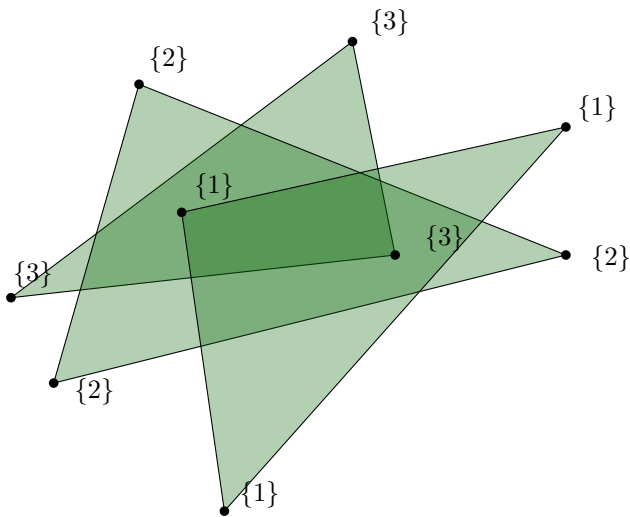
Assign to each vertex a label in $[q]$.



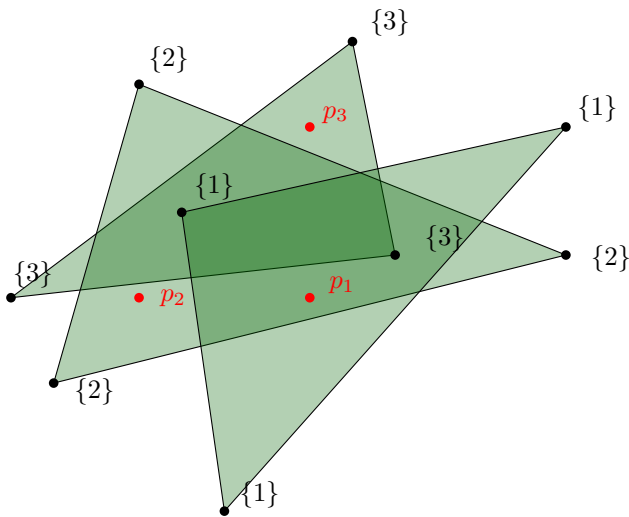
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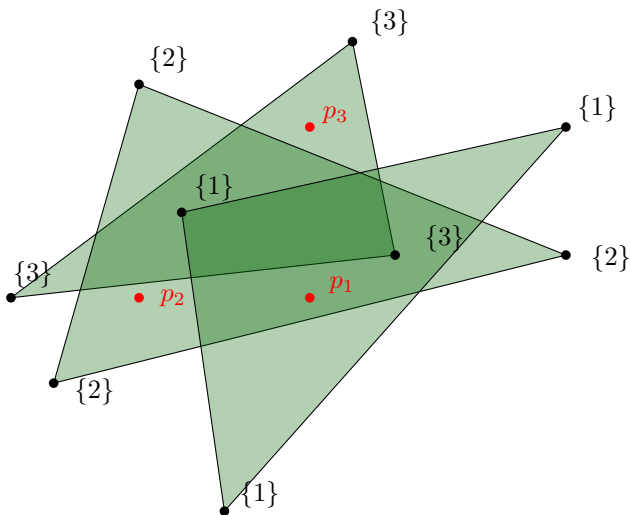
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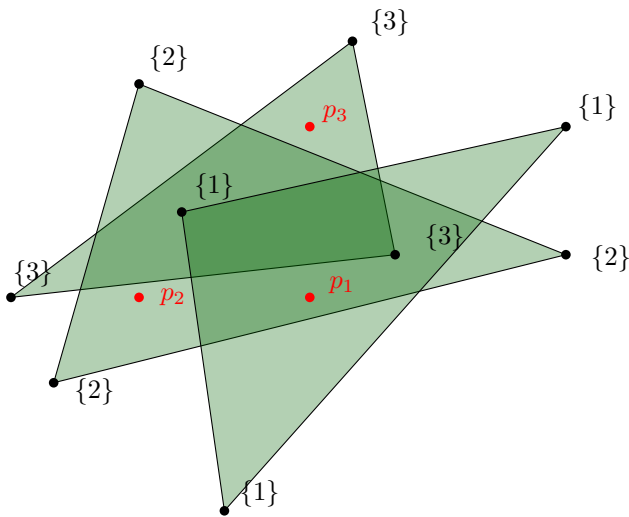
The space of all possible partitions
can be parametrized with $[q]^{*n}$



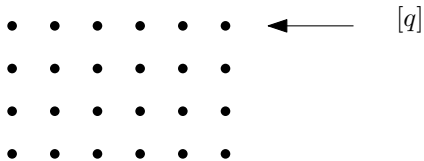
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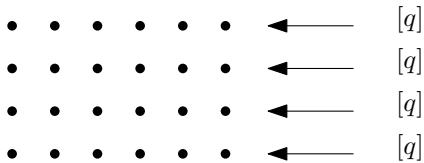


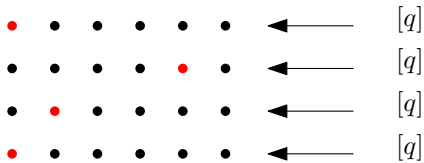
We get a map $[q]^{*n} \rightarrow (\mathbb{R}^{d+1})^q$

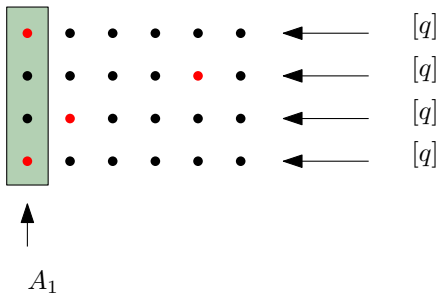


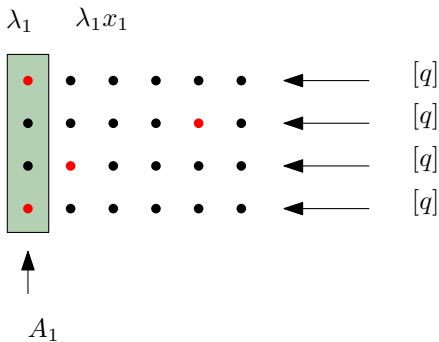
We get a map $[q]^{*n} \rightarrow (\mathbb{R}^{d+1})^q$
 Do we ever have $(p_1, p_2, p_3) = (x, x, x)$ for some x ?





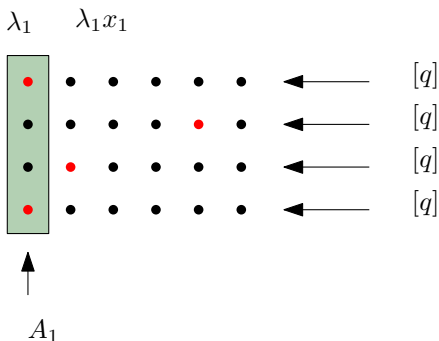






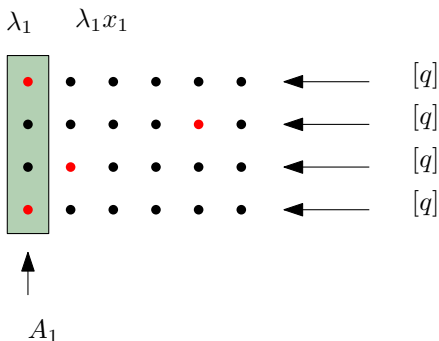
$$(\lambda_1, \lambda_1 f(x_1), \lambda_2, \lambda_2 f(x_2), \dots, \lambda_q, \lambda_q f(x_q))$$

We have a function $\tilde{f} : [q]^{*n} \rightarrow (\mathbb{R}^{d+1})^q$



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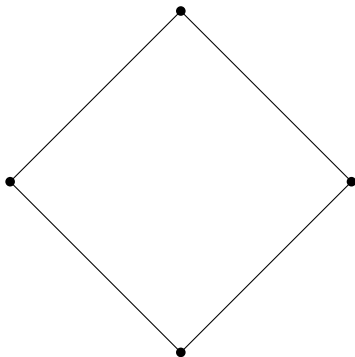
(Dold 1983)

If X and Y have free actions of a group G , X is at least n -connected, and Y is at most n -dimensional, then there exist no continuous equivariant map $X \rightarrow_G Y$.

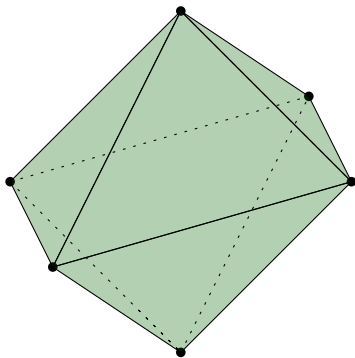
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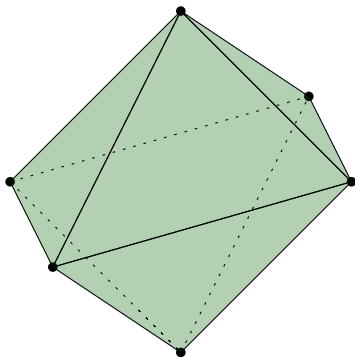
[2]



$$[2]^{*2}$$

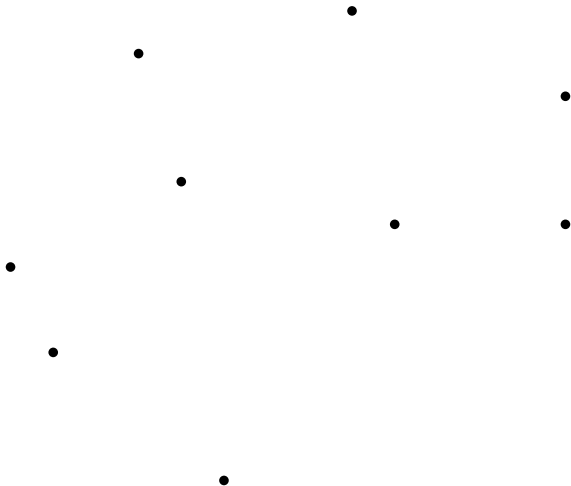


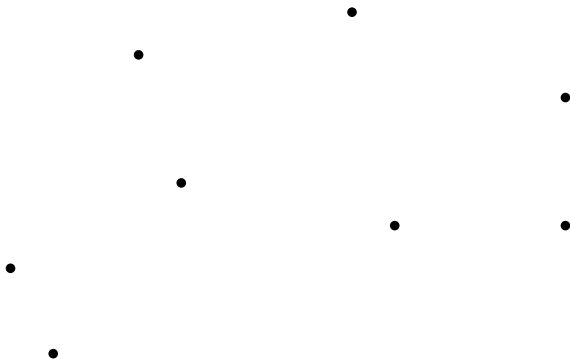
$$[2]^*{}^3$$



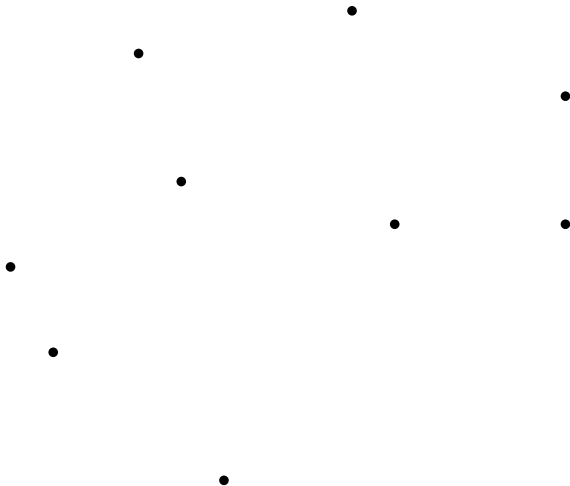
$$[2]^{*3}$$

$[q]^{*n}$ is Highly connected.

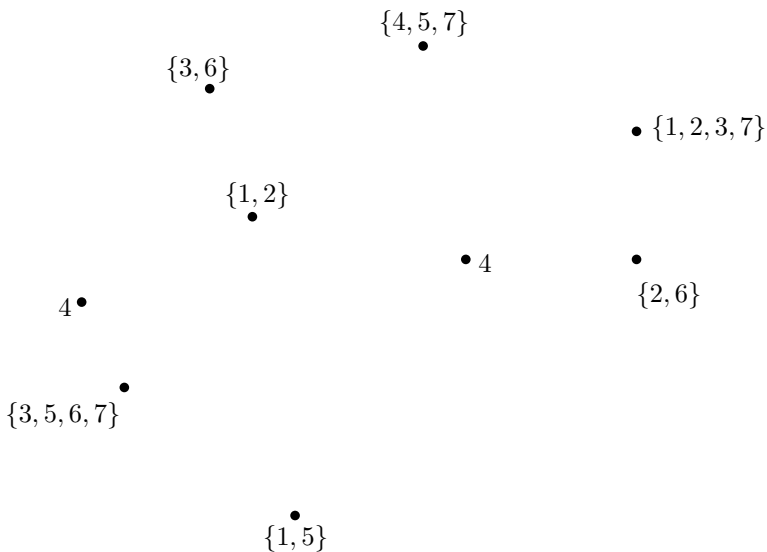




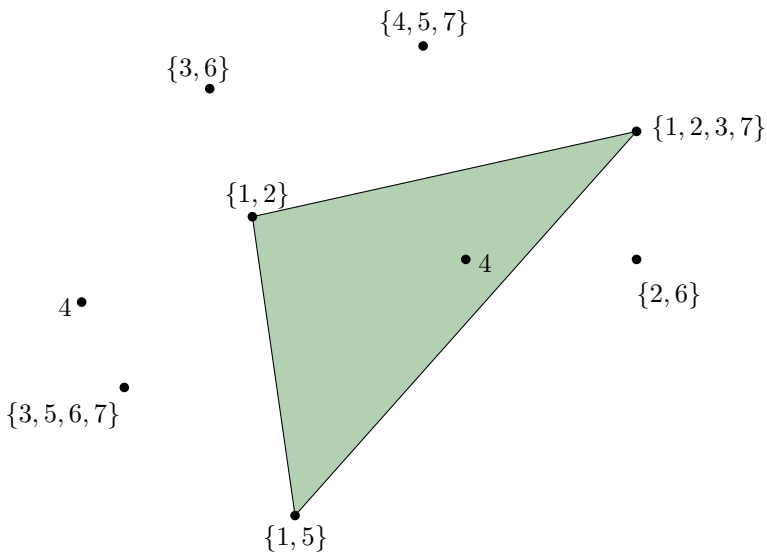
Let p be a very large prime number.



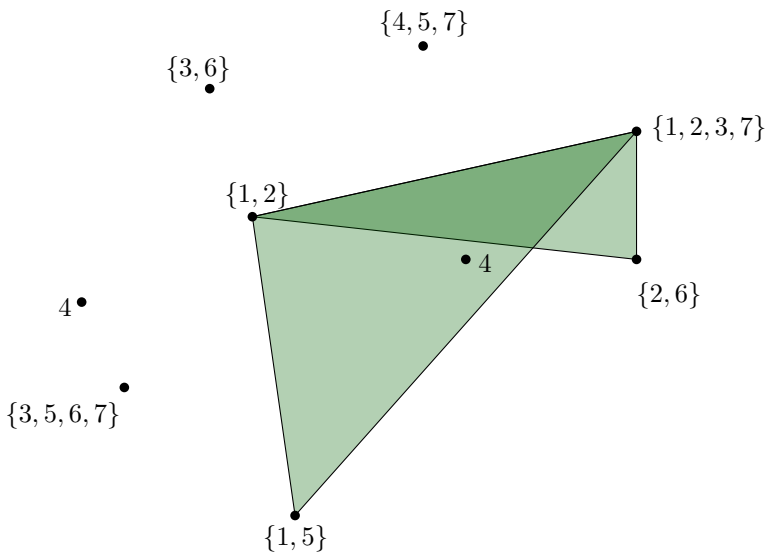
Let p be a very large prime number.
Assign to each vertex a set of labels in $[p]$.



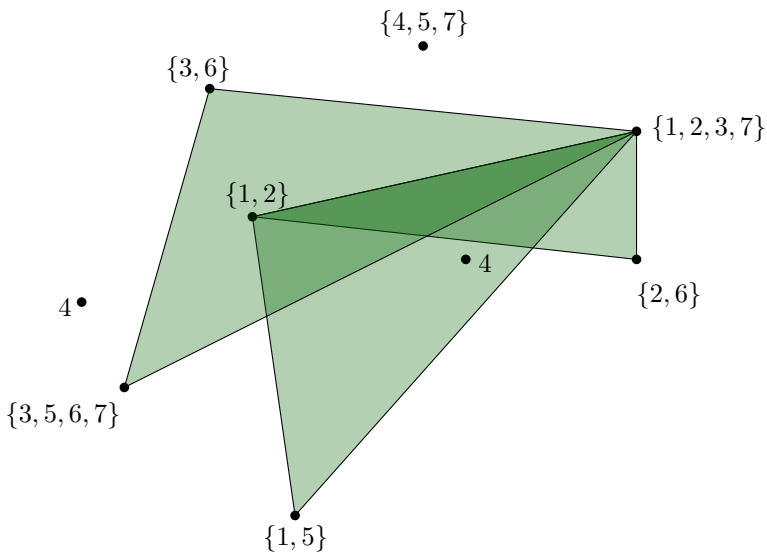
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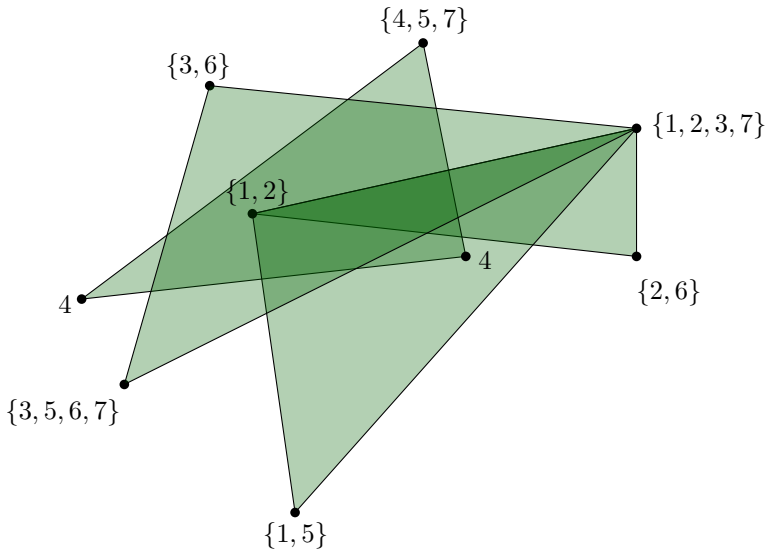
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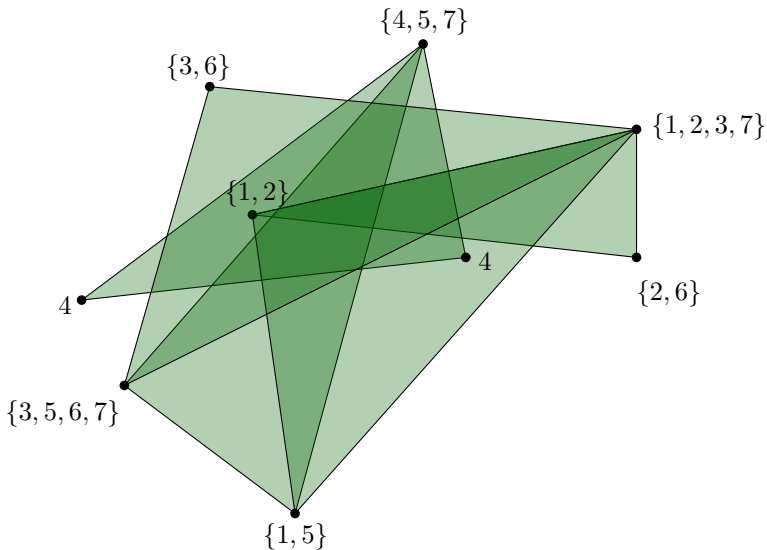
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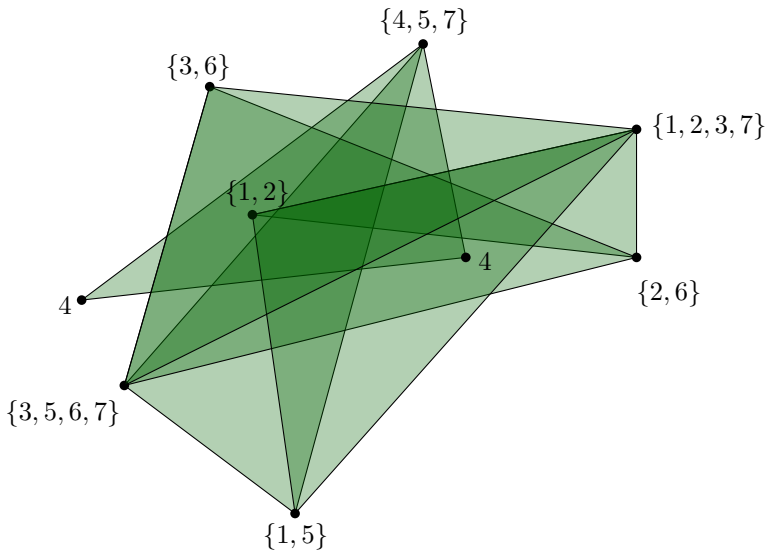
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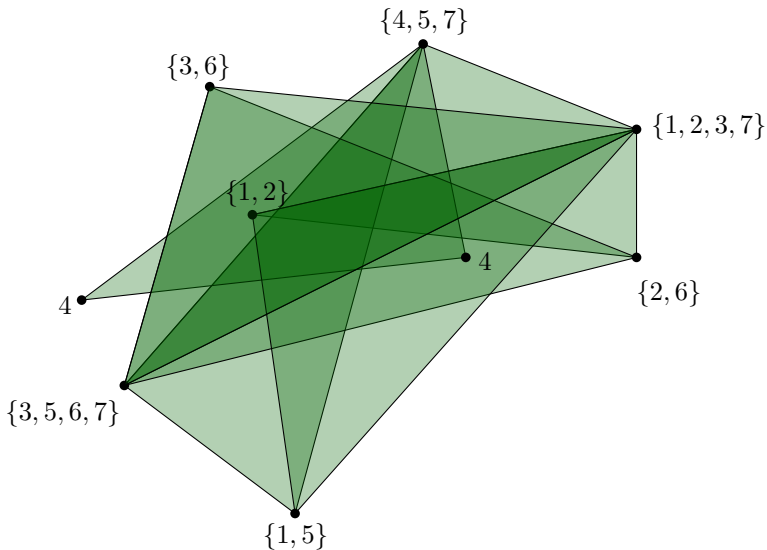
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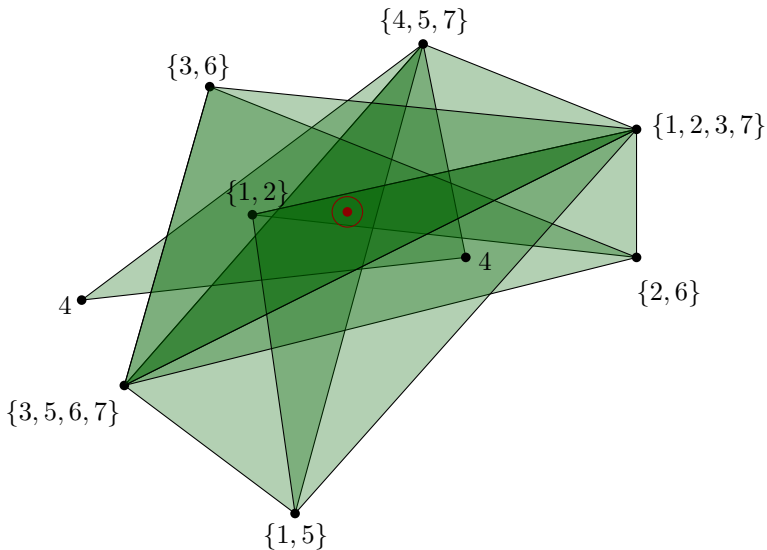
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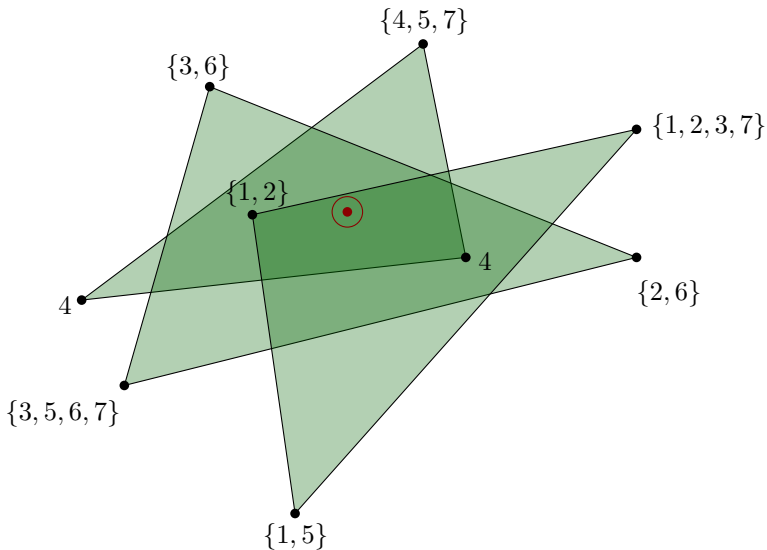
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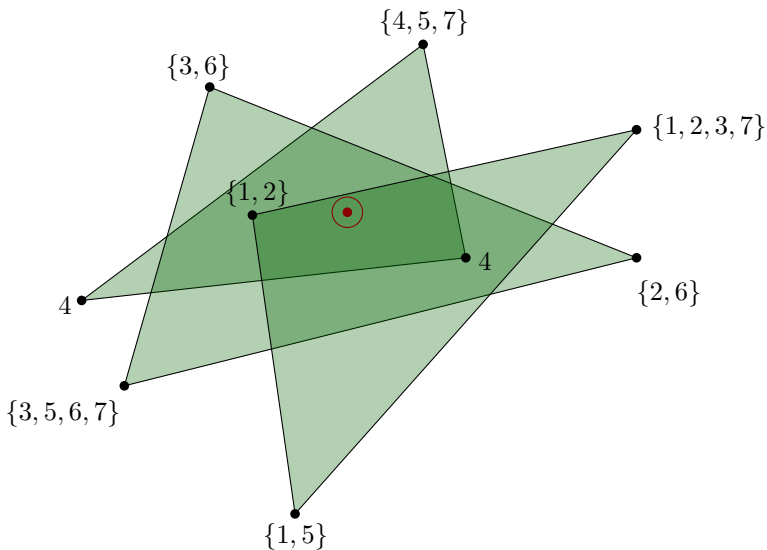
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Assign to each vertex a face of a simplicial complex Σ .

New configuration space: Σ^{*n}

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- ▶ Σ must have a free action of Z_p .

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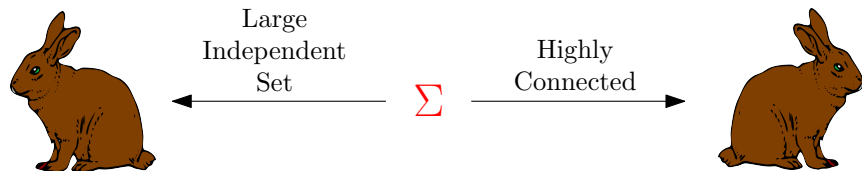
- ▶ Σ must have a free action of Z_p .
- ▶ Σ must be highly connected.

New configuration space: Σ^{*n}

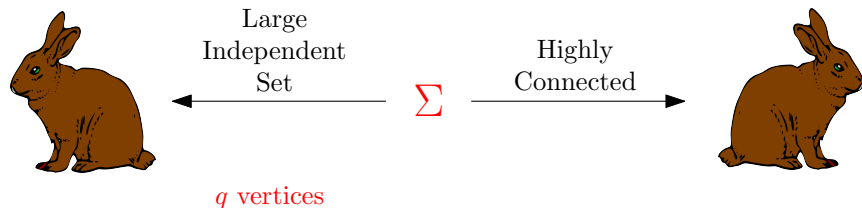
- ▶ Σ must have a free action of Z_p .
- ▶ Σ must be highly connected.
- ▶ Σ must have a large independent set.

Sparse, highly connected, symmetric

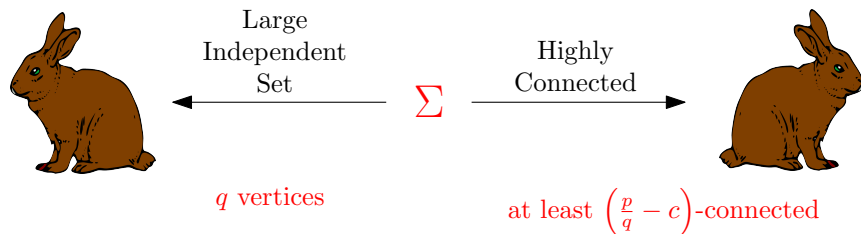
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$$f : \Sigma^{*n} \rightarrow \mathbb{R}^{p(d+1)}$$

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Connectedness < Dimension

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$$n \left(\frac{p}{q} - c + 2 \right) - 2 < \text{Dimension}$$

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$$n \leq q(d+1)$$

$$f : \Sigma^{*n} \rightarrow \mathbb{R}^{p(d+1)}$$

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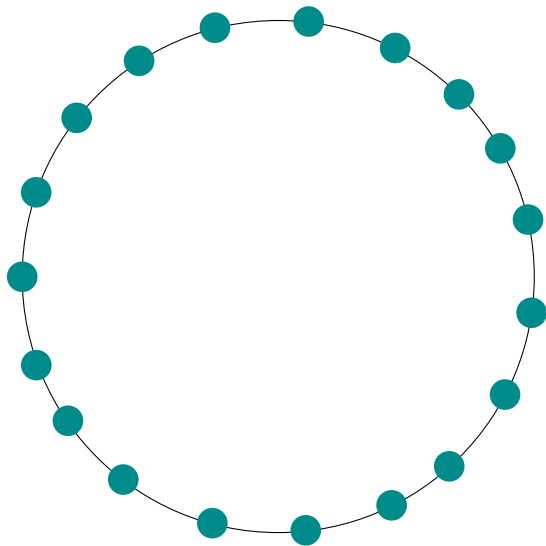
$$f : \Sigma^{*n} \rightarrow \mathbb{R}^{p(d+1)}$$

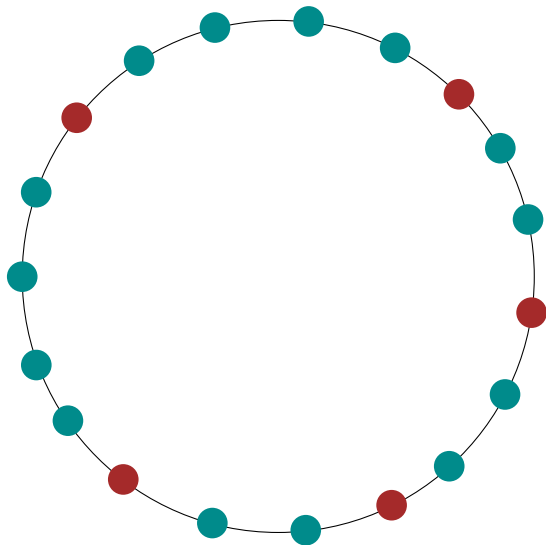
$$n \leq q(d+1)$$

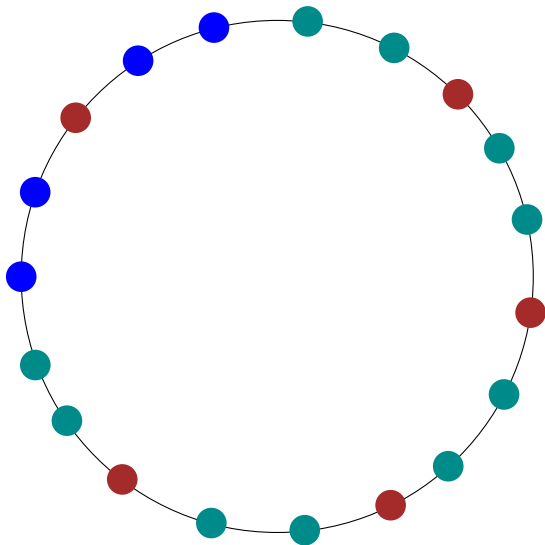
This proves the theorem if $n \geq q(d+1) + 1$

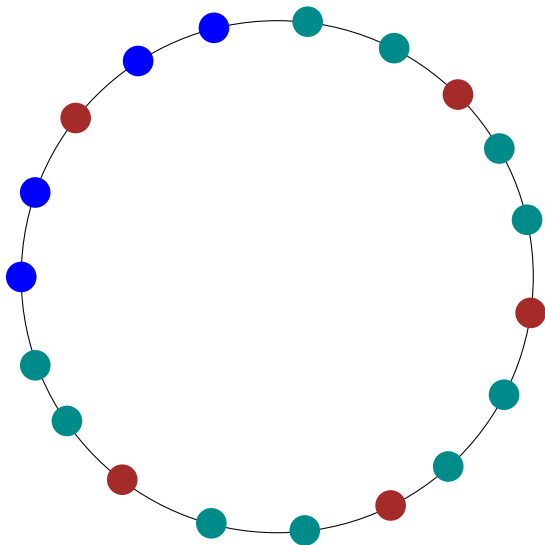
Construction of Σ

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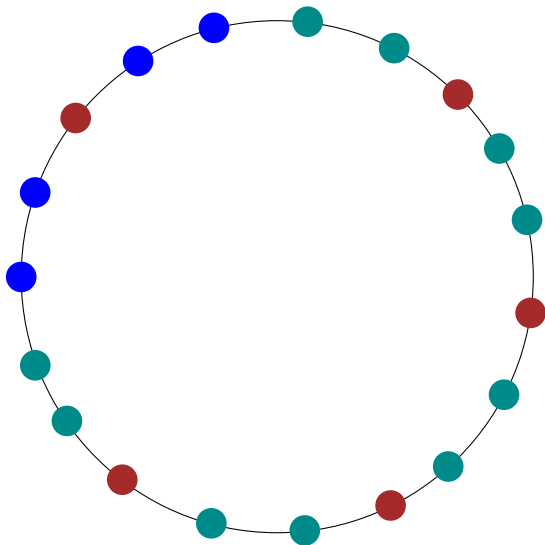




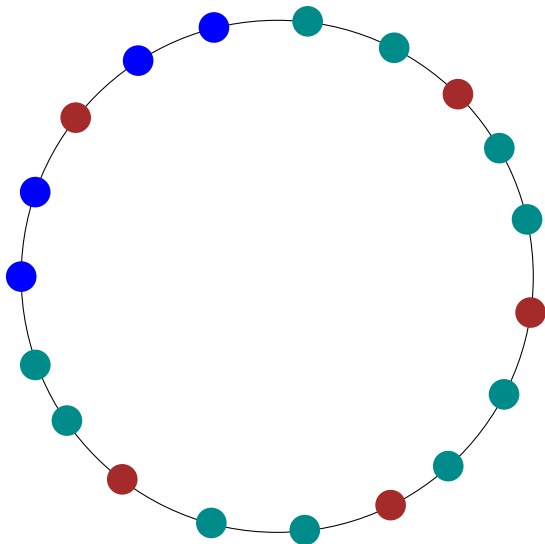




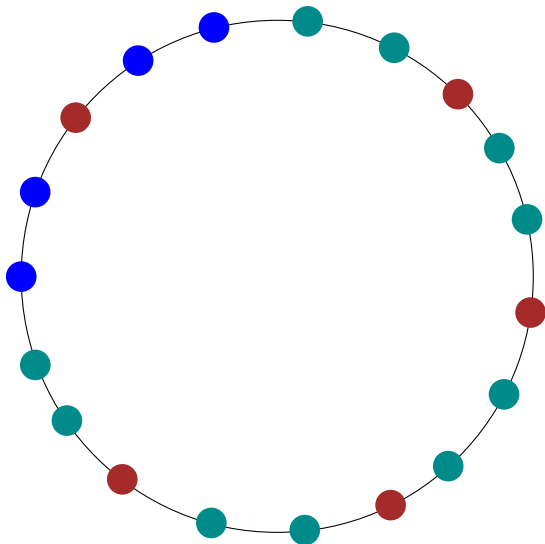
Gaps have at least $q - 1$ vertices



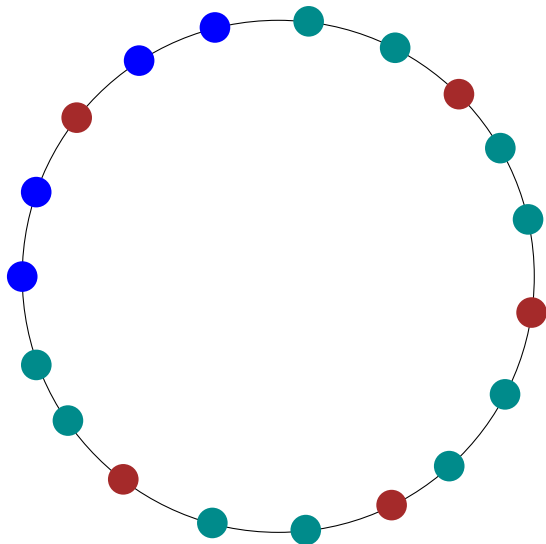
This complex is not connected enough!



It has maximal faces of dimension $\sim \frac{p}{2q-1}$



Idea: Take the faces of dimension $\sim \frac{p}{q}$ and their subsets.



This allows us to prove a topological Tverberg theorem
with $q(d + 1) + 1$ vertices

C_p^a - Subsets of $[p]$ that can be extended to a face with at least a vertices

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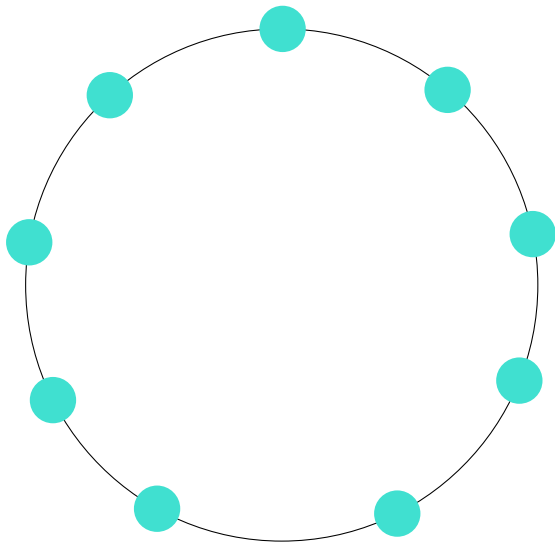
L_p^a - Subsets of $[p]$ that can be extended to a face with at least a vertices

C_p^a - Subsets of $[p]$ that can be extended to a face with at least a vertices - **cyclic**

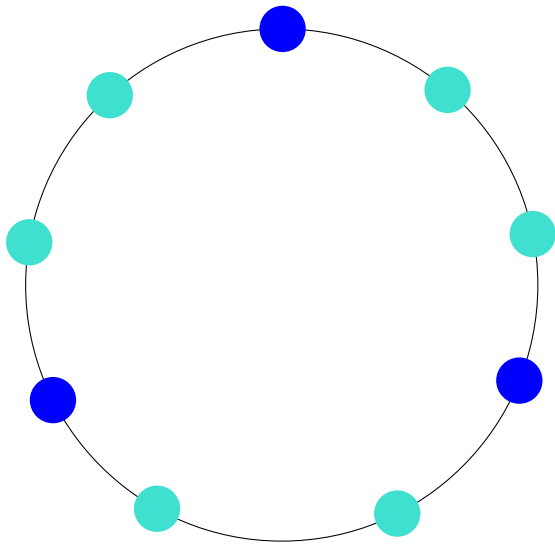
L_p^a - Subsets of $[p]$ that can be extended to a face with at least a vertices - **Linear**

$$C_9^3, q = 3$$

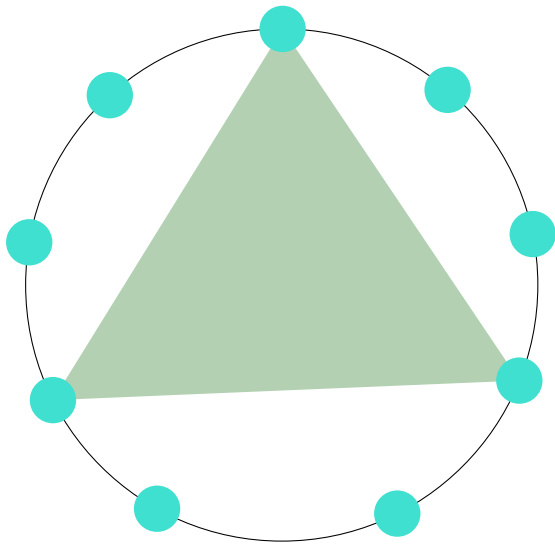
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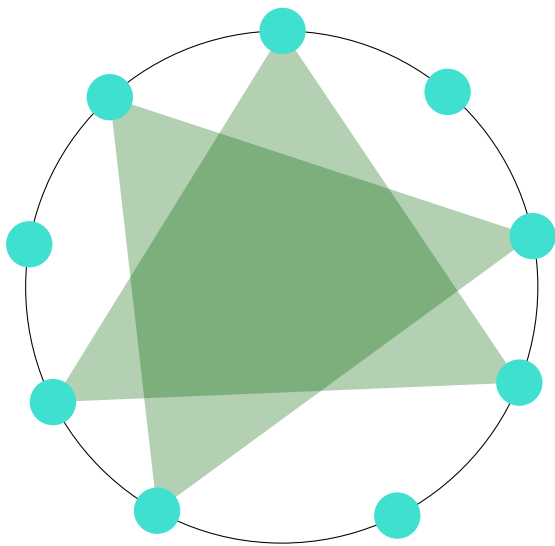
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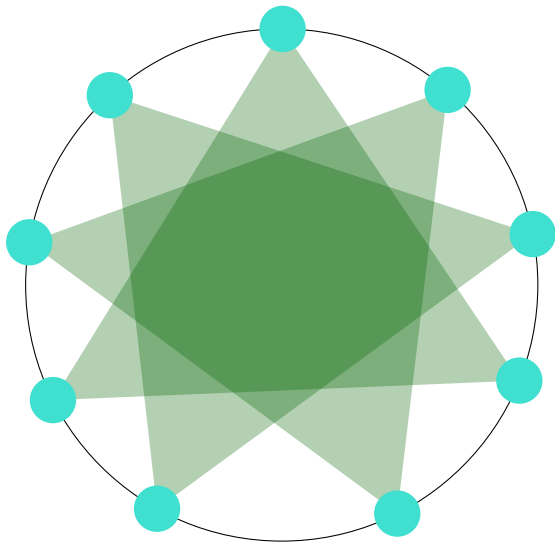
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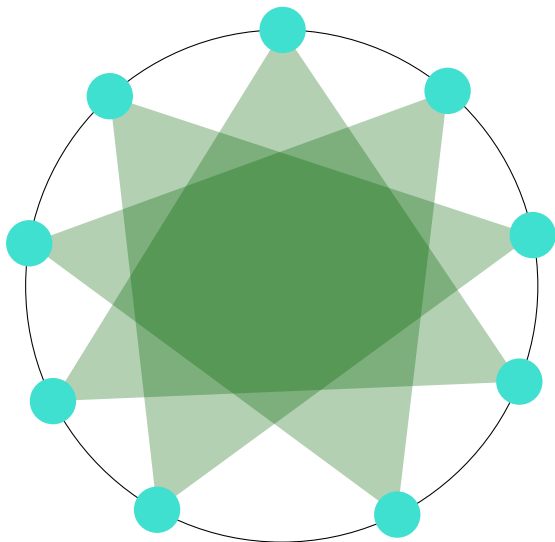
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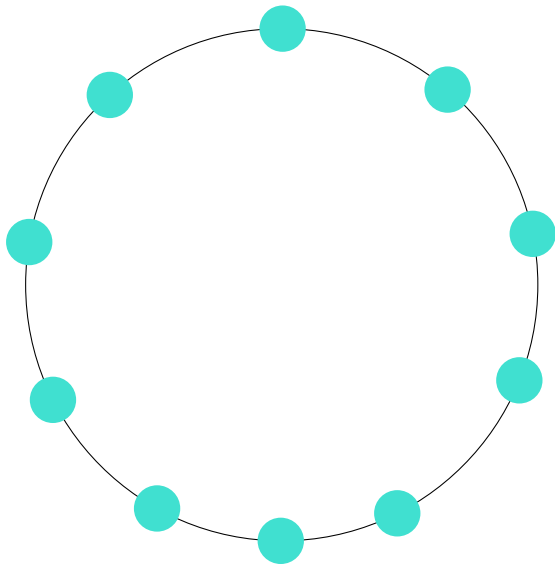
$$C_9^3, q = 3$$



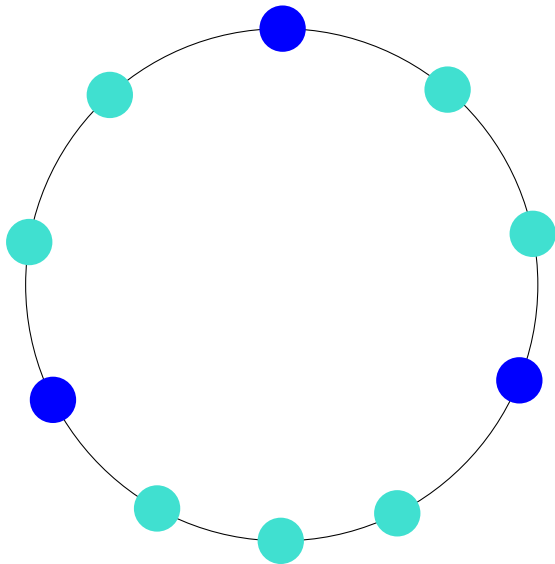
C_{aq}^a is the union of a disjoint simplices.

$$C_{10}^3, q = 3$$

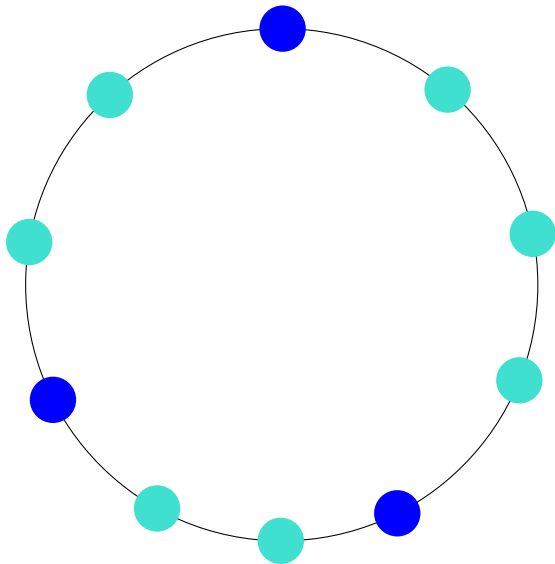
$$C_{10}^3, q = 3$$



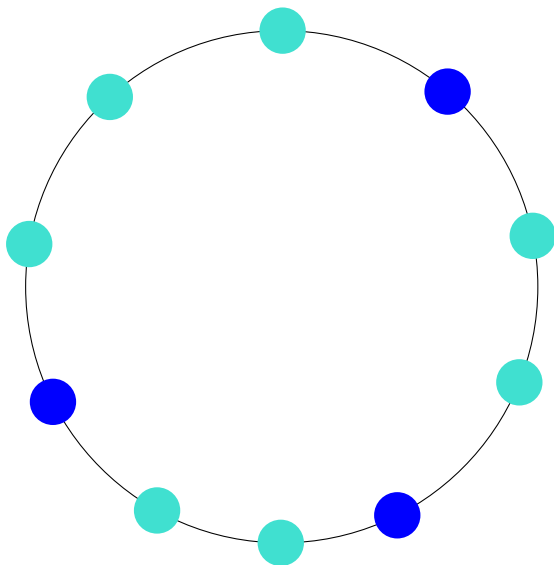
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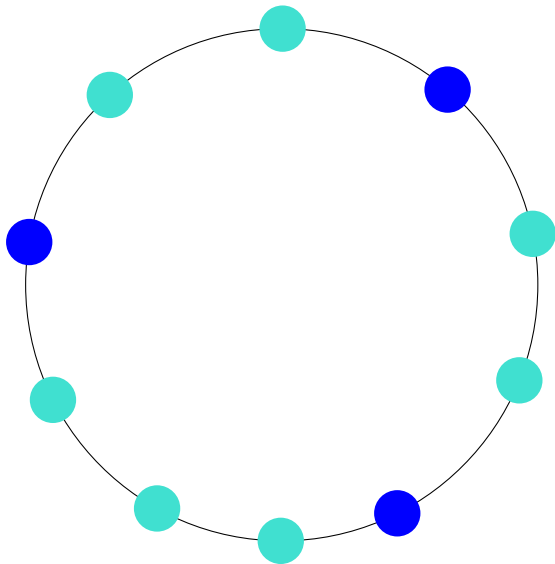
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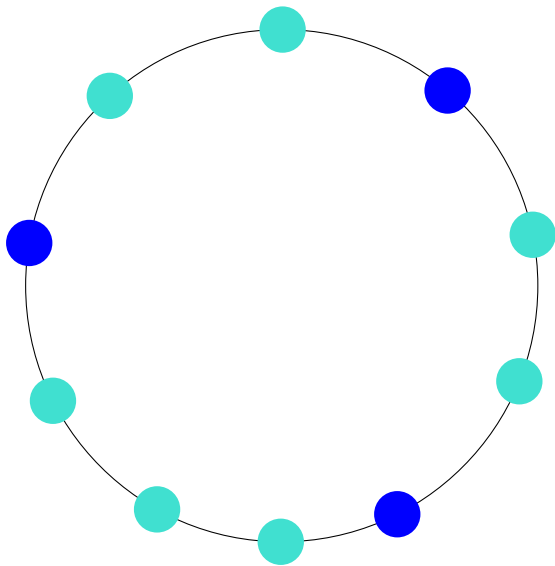
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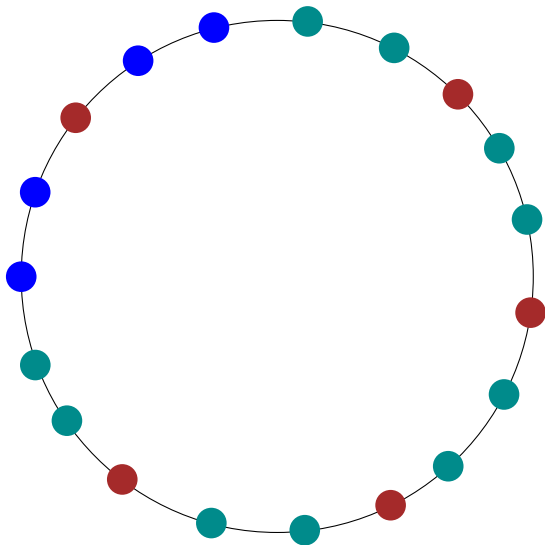


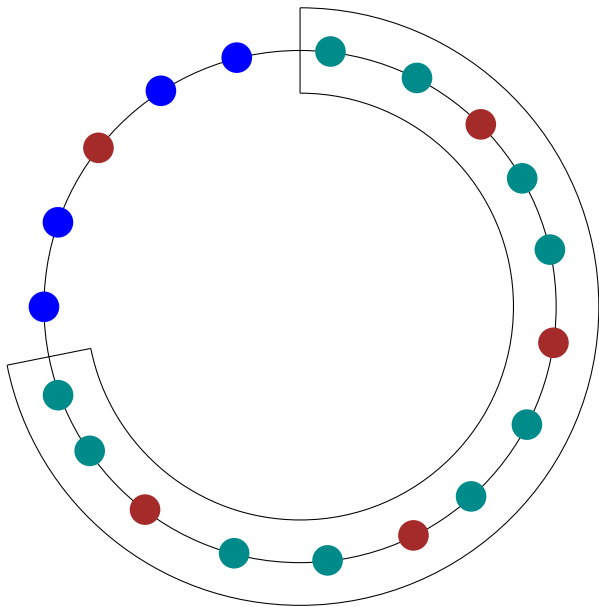
$$C_{10}^3, q = 3$$

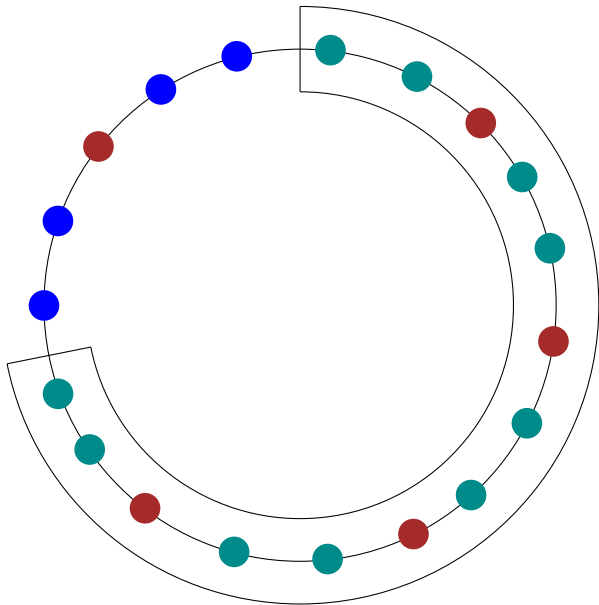


C_{aq+1}^a is a triangulation of a disk bundle.

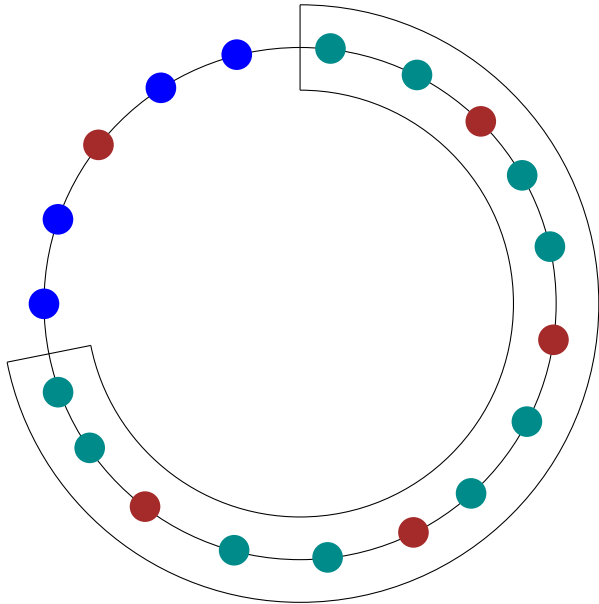
Theorem. $C_{(a+1)q+1}^a$ is at least $(a-2)$ -connected.





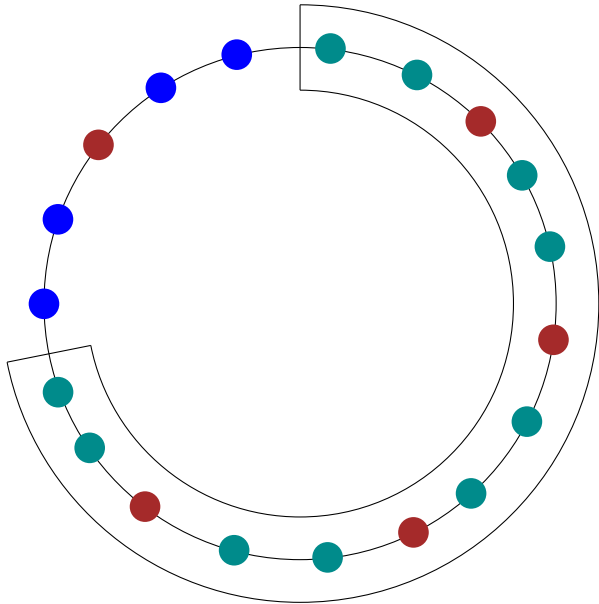


$$C_p^a \mapsto L_{p-2q+1}^{a-1}$$



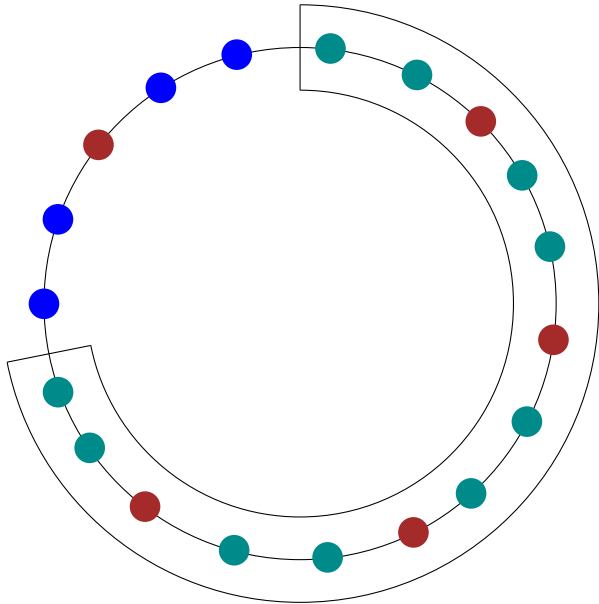
$$C_p^a \mapsto L_{p-2q+1}^{a-1}$$

L_s^r is {



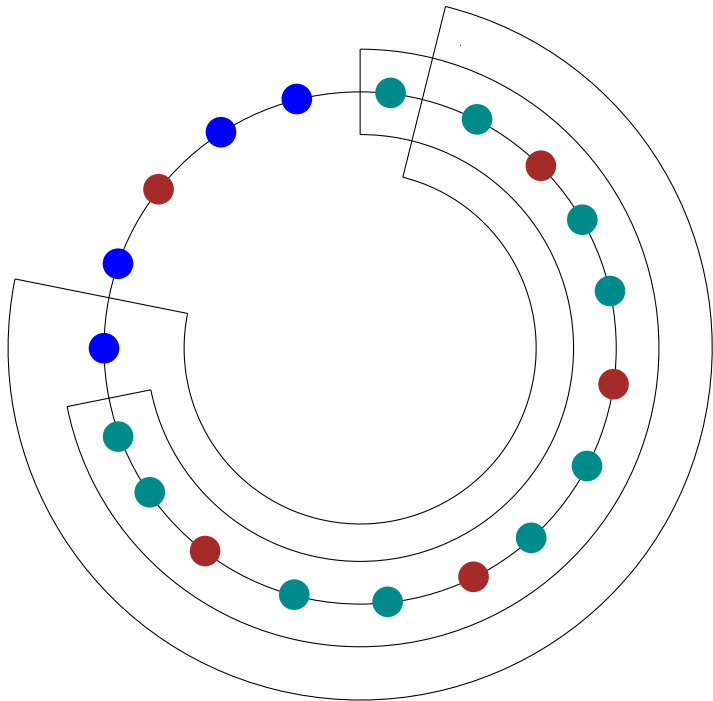
$$C_p^a \mapsto L_{p-2q+1}^{a-1}$$

L_s^r is $\left\{ \begin{array}{l} \text{empty or} \end{array} \right.$



$$C_p^a \mapsto L_{p-2q+1}^{a-1}$$

L_s^r is $\begin{cases} \text{empty or} \\ (r-2) - \text{connected} \end{cases}$



A final application of the german trick finishes the proof.

