Optimal spherical configurations, majorization and f–designs

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Majorization

Let $A = (a_1, \ldots, a_n)$ be an arbitrary sequence of real numbers. $A_\uparrow = (a_{(1)}, \ldots, a_{(n)})$ denote a permutation of elements of A in increasing order: $a_{(1)} \leq a_{(2)} \leq \ldots \leq a_{(n)}$.

$A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$.

A majorizes B, $A \rhd B$, if for all $k = 1, \ldots, n$

$$a_{(1)} + \ldots + a_{(k)} \geq b_{(1)} + \ldots + b_{(k)}.$$

Remark. In *A. W. Marshall and I. Olkin,* **Inequalities: Theory of Majorization and Its Application** it is called a *weak majorization*.

Jensen's inequlity

$$A := (a_1, \ldots, a_m), \quad a_i \in \mathbb{R}$$

$$\bar{A} = (\bar{a}, \ldots, \bar{a}), \quad \text{where} \quad \bar{a} := \frac{a_1 + \ldots + a_m}{m}$$

We have $\bar{A} \rhd A$.

If $s \geq \bar{a}$, then

$$(s, \ldots, s) \rhd (\bar{a}, \ldots, \bar{a}) \rhd (a_1, \ldots, a_m)$$

Jensen's inequality -I

Let f be a **convex** function. Then

$$\frac{f(a_1) + \ldots + f(a_m)}{m} \geq f(\bar{a}).$$

Jensen's inequality – II

Let $s \geq (a_1 + \ldots + a_m)/m$. Then for every **convex and decreasing** function f:

$$\frac{f(a_1) + \ldots + f(a_m)}{m} \geq f(s).$$

The majorization (or Karamata) inequality

Theorem. *Let $f(x)$ be a convex and decreasing function. If $A \triangleright B$ then we have*

$$f(a_1) + \ldots + f(a_n) \leq f(b_1) + \ldots + f(b_n).$$

Moreover, $A \triangleright B$ if and only if for all convex decreasing functions g we have

$$g(a_1) + \ldots + g(a_n) \leq g(b_1) + \ldots + g(b_n).$$

Potential energy E_f

Let S be an arbitrary set. Let $\rho : S \times S \to D \subset \mathbb{R}$ be any symmetric function. Then for a given convex decreasing function $f : D \to \mathbb{R}$ and for every finite subset $X = \{x_1, \ldots x_m\}$ of S we define the potential energy $E_f(X)$ as

$$E_f(X) := \sum_{1 \leq i < j \leq m} f(\rho(x_i, x_j)).$$

Generalized Thomson's Problem

Generalized Thomson's Problem. *For given S, ρ, f and m find all $X \subset S$ with $|X| = m$ such that $E_f(X)$ is the minimum of E_f over the set of all m-element subsets of S.*

The majorization theorem for potentials

$$R_\rho(X) := \{\rho(x_1, x_2) \ldots, \rho(x_1, x_m), \ldots, \rho(x_{m-1}, x_m)\}.$$

Theorem

Let X and Y be two m–subsets of S. Suppose $R_\rho(X) \rhd R_\rho(Y)$. Then for every convex decreasing function f we have $E_f(X) \leq E_f(Y)$.

M and M_f – sets

Definition

We say that $X \in S^m = S \times \ldots \times S$ *is an* M-set in S with respect to ρ *if for any* $Y \in S^m$ *we have that either* $R_\rho(X) \rhd R_\rho(Y)$, *or* $R_\rho(X)$ *and* $R_\rho(Y)$ *are incomparable. Let* $M(S, \rho, m)$ *denote the set of all* M*–sets in* S *of cardinality* m.

Definition

Let $f : D \to \mathbb{R}$ *be a convex decreasing function. Let* $V_f = \inf_{Y \in S^m} E_f(Y)$. *Let* $M_f(S, \rho, m)$ *denote the set of all* $X \in S^m$ *such that* $E_f(X) = V_f$.

M and M_f – sets

Theorem

Let S be a compact topological space and $\rho : S \times S \to D \subset \mathbb{R}$ be a symmetric continuous function. Let $f : D \to \mathbb{R}$ be a strictly convex decreasing function. Then $M_f(S, \rho, m)$ is non-empty and $M_f(S, \rho, m) \subseteq M(S, \rho, m)$.

Theorem

Let $\rho : S \times S \to D \subset \mathbb{R}$ be a symmetric function and $h : D \to \mathbb{R}$ be a convex increasing function. Then $M(S, \rho, m) \subseteq M(S, h(\rho), m)$.

Riesz potential

Let $X = \{p_1, \ldots, p_m\}$ be a subset of $\mathbb{S}^{n-1}$ that consists of distinct points. Then the *Riesz t-energy* of X is given by

$$E_t(X) := \sum_{i<j} \frac{1}{||p_i - p_j||^t}, t > 0, \quad E_0(X) := \sum_{i<j} \log\left(\frac{1}{||p_i - p_j||}\right).$$

Corollary

Let $t \geq 0$. If $X \subset \mathbb{S}^{n-1}$ gives the minimum of E_t in the set of all m-subsets of $\mathbb{S}^{n-1}$, then $X \in M(\mathbb{S}^{n-1}, r_s, m)$ for all $s > -t$.

Minimums of the Riesz potential

Note that for $t = 0$ minimizing E_t is equivalent to maximizing $\prod_{i \neq j} ||p_i - p_j||)$, which is Smale's 7^{th} problem. For $t = 1$ we obtain the Thomson problem, and for $t \to \infty$ the minimum Riesz energy problem transforms into the Tammes problem.

Corollary

Let $t \geq 0$. If $X \subset \mathbb{S}^{n-1}$ gives the minimum of E_t in the set of all m-subsets of $\mathbb{S}^{n-1}$, then $X \in M(\mathbb{S}^{n-1}, r_s, m)$ for all $s > -t$.

$S = \mathbb{S}^{n-1} \subset \mathbb{R}^n$

$x, y \in \mathbb{S}^{n-1}$, $r(x,y) = ||x-y||$, $\varphi(x,y) = 2\arcsin(||x-y||/2)$.

Definition

For any $s \in \mathbb{R}$ denote

$$r_s(x,y) := \begin{cases} r^s(x,y), & s > 0 \\ \log r(x,y), & s = 0 \\ -r^s(x,y), & s < 0 \end{cases}$$

Corollary

(i) $M(\mathbb{S}^{n-1}, r_s, m) \subset M(\mathbb{S}^{n-1}, r_t, m)$ for all $s \leq t$;
(ii) $M(\mathbb{S}^{n-1}, r_s, m) \subset M(\mathbb{S}^{n-1}, \varphi, m)$ for all $s \leq 1$.

$M(\mathbb{S}^1, \varphi, m)$

Theorem

$M(\mathbb{S}^1, \varphi, m)$ consists of regular polygons with m vertices.

This theorem implies that $M(\mathbb{S}^1, r_1, m)$ consists of regular polygons.

However, the set $M(\mathbb{S}^1, r_2, m)$, $m \geq 4$, is much larger. In fact, $M(\mathbb{S}^1, r_2, 4)$ consists of quadrilaterals with sides (in angular measure) $(2\pi - 3\alpha, \alpha, \alpha, \alpha)$, where $\pi/2 \leq \alpha \leq 2\pi/3$.

Optimality of regular simplices

Theorem

Let $s \leq 2$. Then $M(\mathbb{S}^{n-1}, r_s, n+1)$ consists of regular simplices.

Open problem. It is easy to see that
$M(\mathbb{S}^{n-1}, \varphi, n+1) \neq M(\mathbb{S}^{n-1}, r_2, n+1)$ for $n \geq 3$.

I think that $M(\mathbb{S}^2, \varphi, 4)$ consists of vertices of tetrahedrons $\Delta_{a,\theta}$ with $a \in [0, 1/\sqrt{3}]$ and $0 < \theta \leq \pi/2$.

Here $\Delta_{a,\theta}$ is a two–parametric family of tetrahedrons $ABCD$ in $\mathbb{S}^2$ such that its opposite edges AC and BD are of the same lengths and the angle between them is θ. Let X be the midpoint of AC and Y be the midpoint of BD. Then X, Y and O (the center of $\mathbb{S}^2$) are collinear. $a = |OX| = |OY|$.

Optimal constrained $(n+k)$–sets

Theorem

Let $2 \leq k \leq n$ and $s \leq 2$. Then $M(\mathbb{B}^n, r_s, \sqrt{2}, n+k) = M(\mathbb{S}^{n-1}, r_s, \sqrt{2}, n+k)$ and this set consists of k orthogonal to each other regular d_i–simplexes S_i such that all $d_i \geq 1$ and $d_1 + ... + d_k = n$.

This theorem follows from the above and Wlodek Kuperberg theorems.

Spherical three–point M–sets

$$(1-t)^z + 2^{z-1}(1-t^2)^z = \left(\frac{3}{2}\right)^{z+1}, \ z = \frac{s}{2}. \qquad (1)$$

For all s this equation has a solution $t = -1/2$. If

$$4 > s \geq s_0 := \log_{4/3}(9/4) \approx 2.8188,$$

then (1) has one more solution $t_s \in (-1, -1/2)$.

$$t_{s_0} = -1, \quad t_4 = -1/2,$$

Spherical three–point M–sets

Theorem

There are three cases for $M := M(\mathbb{S}^1, r_s, 3)$

1. *If* $s \leq \log_{4/3}(9/4)$, *then* M *contains only regular triangles.*
2. *If* $\log_{4/3}(9/4) < s < 4$, *then* M *consists of regular triangles and triangles with central angles* $(\alpha, \alpha, 2\pi - 2\alpha)$, *where* $\alpha \in (\arccos(t_s), \pi]$.
3. *If* $s \geq 4$, *then* M *consists of regular triangles and triangles with central angles* $(\alpha, \alpha, 2\pi - 2\alpha)$, $\alpha \in [2\pi/3, \pi]$.

Spherical four–point M–sets

$M(\mathbb{S}^1, \varphi, 4)$ contains only squares.
Then $M(\mathbb{S}^1, r_s, 4)$ with $s \leq 1$ also contains only squares.

It is an interesting problem to find $M(\mathbb{S}^1, r_s, 4)$ for all s.

It can be proven that $M(\mathbb{S}^1, r_2, 4)$ consists of quadrilaterals inscribed into the unit circle with central angles $(\alpha, \alpha, \alpha, 2\pi - 3\alpha)$, where $\pi/2 \leq \alpha \leq 2\pi/3$.

$M(\mathbb{S}^2, r_s, 4)$ with $s \leq 2$ contains only regular tetrahedrons.

The case $s > 2$ is open?

Spherical five–point M–sets

$M(\mathbb{S}^1, \varphi, 5)$ and $M(\mathbb{S}^1, r_s, 5)$ with $s \leq 1$ contain only regular pentagons.

$M(\mathbb{S}^2, r_s, \sqrt{2}, 5)$, $s \leq 2$, contains only *triangular bi-pyramid* (TBP). The same result holds for $M(\mathbb{S}^2, \varphi, \sqrt{2}, 5)$.

The last known case is $M(\mathbb{S}^3, r_s, 5)$ with $s \leq 2$ that contains only regular 4–simplexes.

It is a very interesting open problem to find $M(\mathbb{S}^2, r_s, 5)$.

For any t the global minimizer of the Riesz energy R_t of 5 points lies in $M(\mathbb{S}^2, r_s, 5)$ for any s.

It is proved that the TBP is the minimizer of R_t for $t = 0$ [Dragnev et al] and for $t = 1, 2$ [Schwartz]. Note that the TBP is not the global minimizer for R_t when $t > 15.04081$

Definition of f-design

$P = \{p_1, \dots, p_m\} \subset \mathbb{S}^{n-1}$. Define the k–th moment of P:

$$M_k(P) := \sum_{i=1}^{m} \sum_{j=1}^{m} G_k^{(n)}(t_{i,j}), \quad t_{i,j} := p_i \cdot p_j = \cos(\varphi(p_i, p_j))$$

$M_k(P) \geq 0$ for all $k = 1, 2, \dots$

Definition

$P = \{p_1, \dots, p_m\} \subset \mathbb{S}^{n-1}$. $D(P) := \{p_i \cdot p_j, i \neq j\}$.
$f(t) = \sum_k f_k \, G_k^{(n)}(t)$. *$P$ is an f–design if*

1. *For all $k > 0$ with $f_k \neq 0$ we have $M_k(P) = 0$;*
2. *$D(P) \subset Z_f$, where $Z_f := \{t \in [-1, 1) | f(t) = 0\}$.*

We say that an f-design is of degree d *if* deg $f = d$.

Delsarte's bound and f–designs

Lemma

Let $f(t) = \sum_k f_k \, G_k^{(n)}(t) \in C([-1,1])$*. If there is an* f*–design in* $\mathbb{S}^{n-1}$ *of cardinality* m*, then* $f(1) = mf_0$*.*

Theorem

Let $f(t) = \sum_k f_k \, G_k^{(n)}(t) \in C([-1,1])$ *with all* $f_k \geq 0$*. Let* $P \subset \mathbb{S}^{n-1}$ *with* $|P| = m$ *is such that* $D(P) \subset Z_f$*. Then* P *is an* f*–design if and only if* $f(1) = mf_0$*.*

Spherical f-designs and M–sets

Theorem

Let $f(t) = \sum_k f_k \, G_k^{(n)}(t)$ be a function on $[-1, 1]$ with all $f_k \geq 0$. Then any f–design in $\mathbb{S}^{n-1}$ is an M–set with $\rho(x, y) = -f(x \cdot y)$.

Open problem. Consider f with all $f_k \geq 0$ and $f(1) = mf_0$. By the theorem, if $D(P) \subset Z_f$ then P is an f–design and $P \in M(\mathbb{S}^{n-1}, -f, m)$.
It is easy to prove that if $Y \in M(\mathbb{S}^{n-1}, -f, m)$, then $D(Y) \subset Z_f$.

The question: is Y isomorphic to P?

Spherical τ– and f–designs

P is a τ-design if and only if $M_k(P) = 0$ for all $k = 1, 2, \ldots, \tau$

Theorem

If $P \subset \mathbb{S}^{n-1}$ is a τ–design and $|D(P)| \leq \tau$, then P is an f–design of degree τ with

$$f(t) = g(t) \prod_{x \in D(P)} (t - x), \quad \deg g \leq \tau - |D(P)|.$$

Spherical two–distance sets and f–designs

Theorem

Let $f(t) = (t - a)(t - b)$ and $a + b \neq 0$. Then P in $\mathbb{S}^{n-1}$ is an f–design if and only if P is a two–distance 2–design.

If $b = -a$ then f–designs are *equiangular lines sets.*
There is a correspondence between f-designs of degree 2 and *strongly regular graphs.*

Let Λ_n be the set of points $e_i + e_j$, $1 \leq i < j \leq n + 1$ in $\mathbb{R}^{n+1}$.
In fact, Λ_n is a maximal f–design of degree 2. *Are there other maximal f–designs with $a + b > 0$ of degree $d \geq 2$?*

Every graph G can be embedded as a spherical two–distance set.
What graphs can be embedded as f–designs?

THANK YOU