Planar point sets with many similar triangles

Jozsef Solymosi (Joint work with Dhruv Mubayi)

Elekes and Erdős

Elekes and Erdős proved that for any triangle, T, there are n-element planar point sets with $O(n^2)$ triangles similar to T.

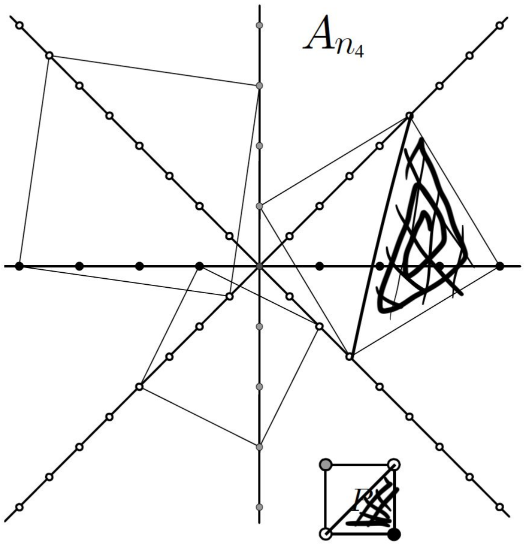
It was proved shortly after that if the number of equilateral triangles is at least $(1/6+\varepsilon)n^2$ then the pointset should contain large parts of a triangular lattice.

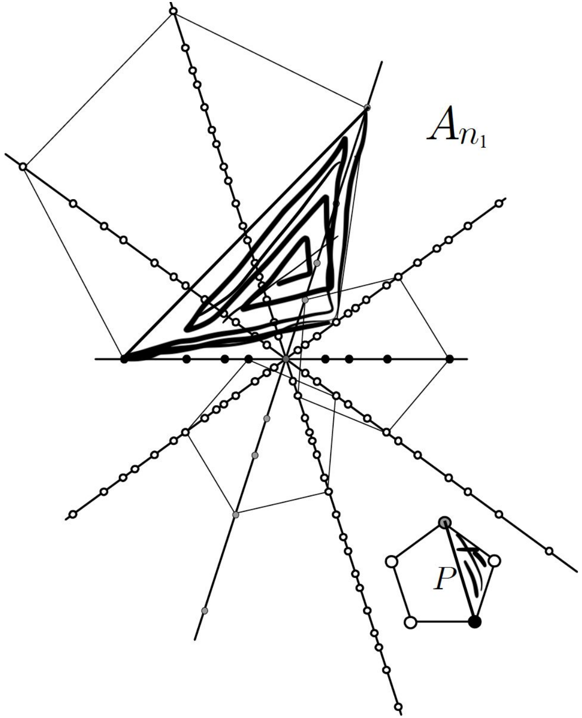
On the other hand, no lattice is guaranteed by cn^2 similar copies if $c<1/6$.

Laczkovich-Ruzsa (1997)

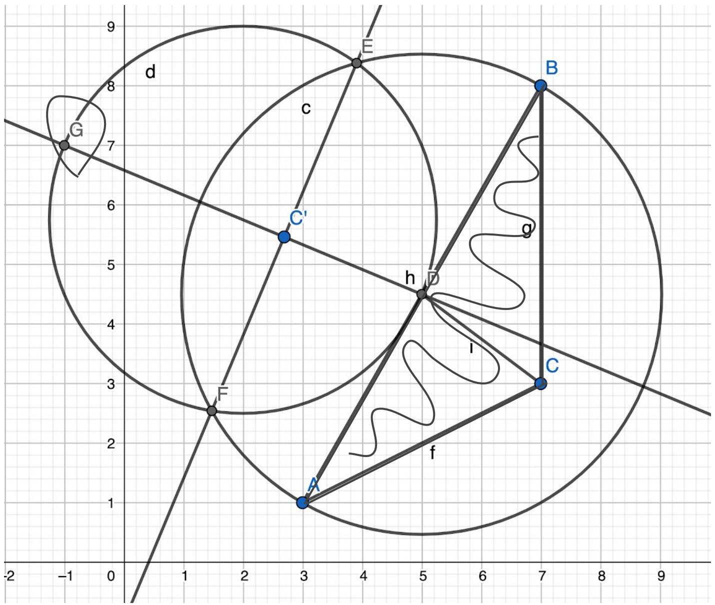
“We investigate the maximal number $S(P, n)$ of subsets of a set of n elements homothetic to a fixed set P. Elekes and Erdős proved that $S(P, n) > cn^2$ if $|P| = 3$ or the elements of P are algebraic. For $|P| \geq 4$ we show that $S(P, n) > cn^2$ if and only if every quadruple in P has an algebraic cross ratio.”

Structural results for planar sets with many similar subsets (Abrego, Elekes, Fernandes-Merchant (2002))





Inversion points of triangles



G is the *inversion point* of the *ABC* triangle.

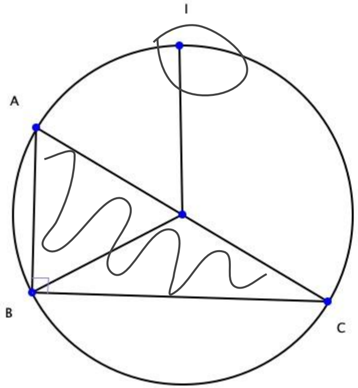
C' is the reflection point of ***C*** about the *AB* side and **G** is the image of ***C'*** after inversion about the circle with diameter *AB*.

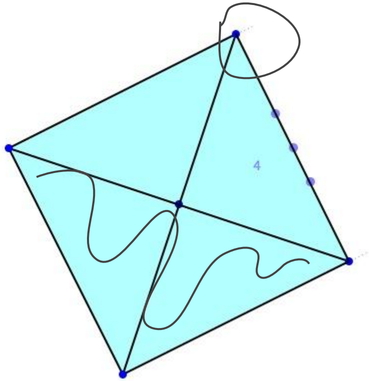
Using complex numbers:

$G = D - (A-B)^2 / 2(A+B-2C)$

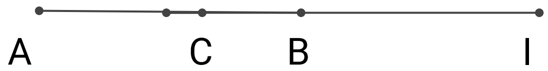
Some special cases

- Right triangles

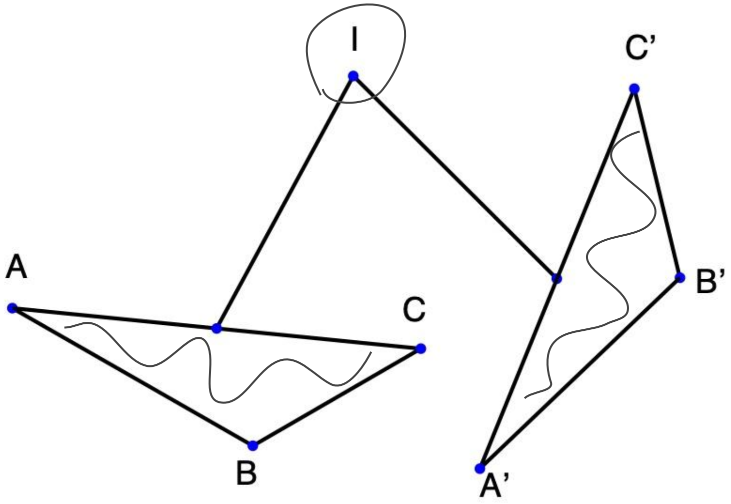




- Linear triples



Structure induced by many similar triangles

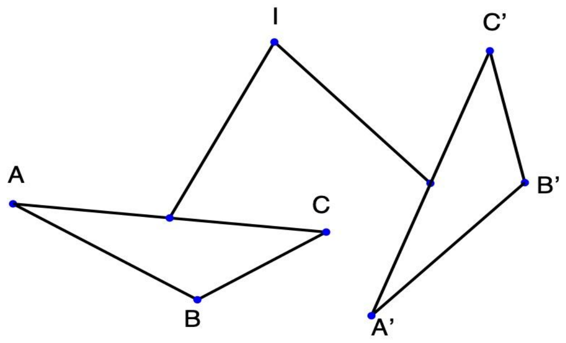


Two triangles sharing their inversion points

I is the common inversion point of two similar triangles

A Key Lemma

Lemma: There is a universal constant, *c>0,* such that the following holds. Let us suppose that *n* points in the plane span at least $n^{11/6}$ triangles, similar to a given triangle *T*. Then for any selection of $n^{11/6}$ triangles similar to *T*, there will be at least two vertex disjoint triangles sharing the same inversion point.



Katz-Tao

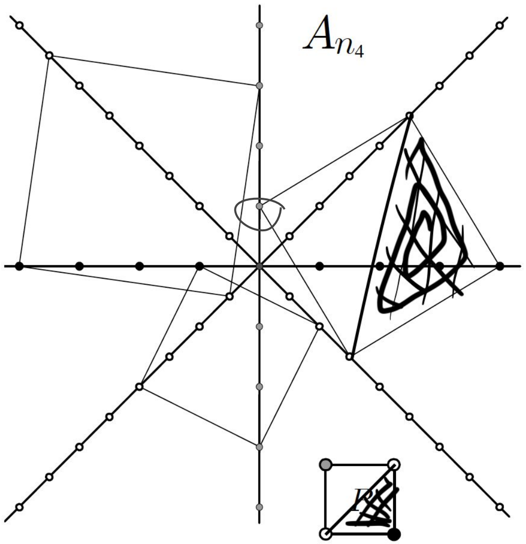
This result on similar triangles is a "geometrization" of a very nice paper by Netz Katz and Terry Tao:

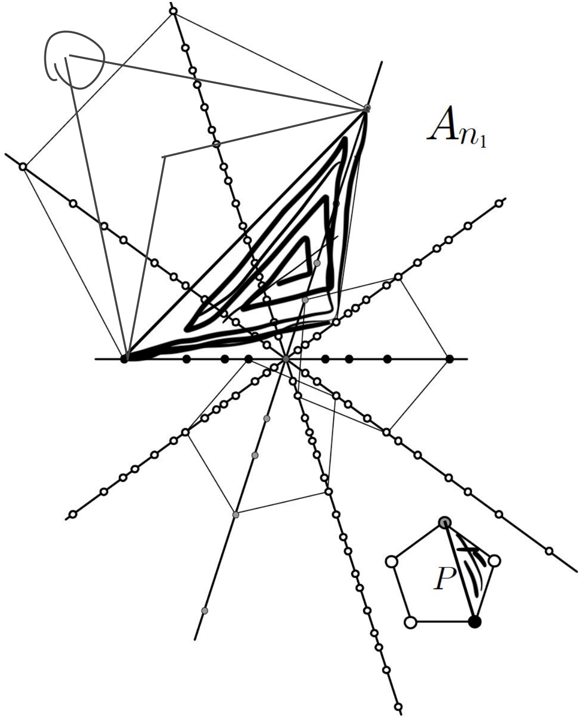
Bounds on arithmetic projections, and applications to the Kakeya conjecture, Mathematical Research Letters, Volume 6 (1999) Pages: 625-630.

"Let A, B, be finite subsets of a torsion-free abelian group, and let $G \subset A \times B$ be such that $\#A, \#B, \#\{a+b : (a,b) \in G\} \leq N$. We consider the question of estimating the quantity $\#\{a-b : (a,b) \in G\}$."

In our settings $a+b$ is vertex C and $a\text{-}b=Inv(C)$.

Structural results for planar sets with many similar subsets (Abrego, Elekes, Fernandes-Merchant (2002))





Pentagons in graphs

Bollobás and Győri: If a graph on n vertices has more than $Cn^{3/2}$ triangles then it contains pentagons.

Füredi and Özkahya: If a three-uniform hypergraph on n vertices has at least $Cn^{3/2}$ edges then it contains a Berge pentagon.

Related questions were studied by Alon and Shikhelman and by Kostochka, Mubayi, and Verstraete, and later by Ergemlidze, Methuku, Salia, and Győri.

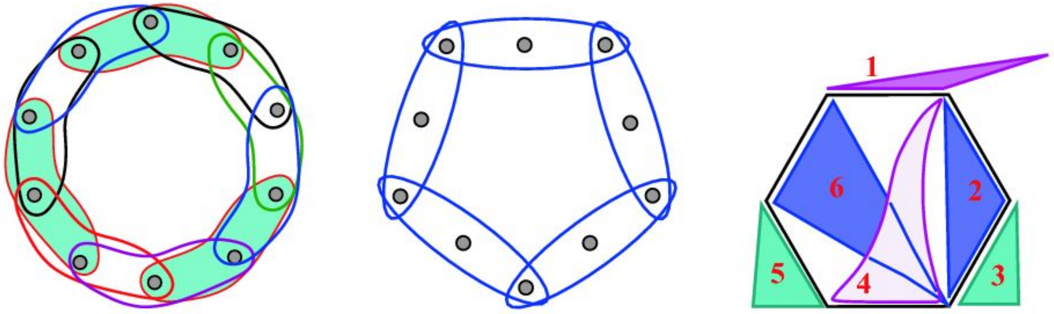
Bárány and Füredi analysed pointsets with many *almost* similar triangles.

Number of pentagons in linear 3-graphs

If a three-uniform linear hypergraph on n vertices has at least m edges, where

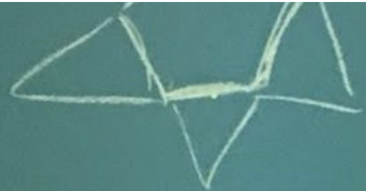
$$n^{3/2} << m << n^2$$

then it contains at least m^6/n^7 (Berge) pentagons.



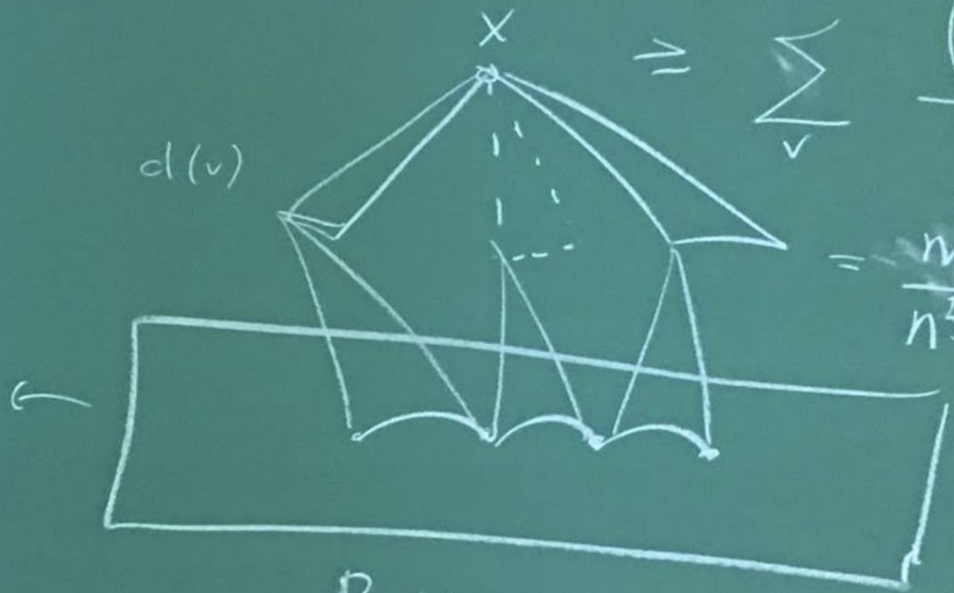
(a) Tight cycle (b) Loose cycle (c) Berge cycle

Picture from Simonovits-Szemerédi, *Embedding Graphs into Larger Graphs: Results, Methods, and Problems*



Step II: $m \gg n$

$$\# C_5 \geq \frac{1}{5} \sum_x P_3(x)$$



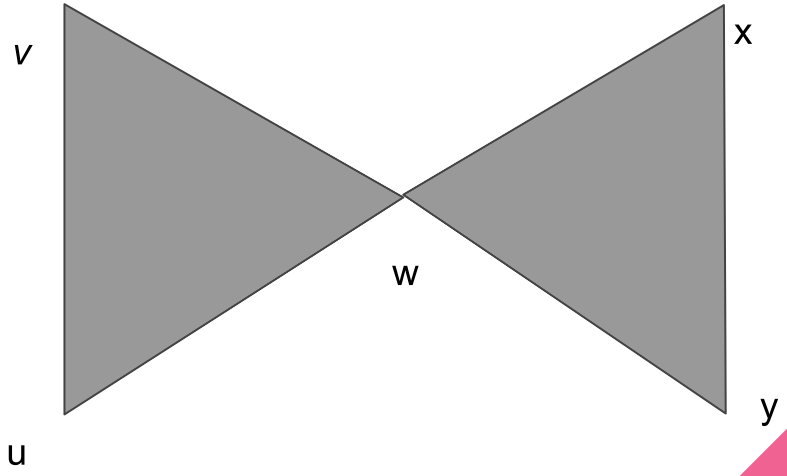
$$\geq \sum_v \frac{\left(\frac{d(v)\cdot m}{n}\right)^3}{n^2} = \frac{m^3}{n^5} \sum d(v)^3 \geq \frac{m^3}{n^5} \cdot n \cdot \left(\frac{m}{n}\right)^3 = \frac{m^6}{n^7}$$

$$|C_3(x)| \geq \boxed{d(v) \cdot \frac{m}{n}}$$

$P_3(x) = \# 3$ paths

Counting Pentagons

Given a three-uniform hypergraph, *H*, and vertex *u*, define the bowtie graph of *u*, written *B(u)*, as the set of pairs *xy* such that there are edges *uvw* and *wxy* in *H*



Let $b(u) = |B(u)|$

If *H* has *m* edges, then by averaging

$m^2/n \leq \Sigma\, b(u)$

3 edge Paths

Let $p_3(u)$ be the number of 3-edge paths in $B(u)$. Then by the Blakley-Roy* inequality

$p_3(u) \geq n\ (2b(u)/n - 3)^3$

$$\#\text{ pentagons in } H \geq \tfrac{1}{5} \Sigma_u\, p_3(u)$$

$$\geq \tfrac{1}{5} \Sigma_u\, n\ (2b(u)/n - 3)^3$$

$$(n \text{ large}) \qquad \geq (1/n^2)\ \Sigma_u\, b(u)^3$$

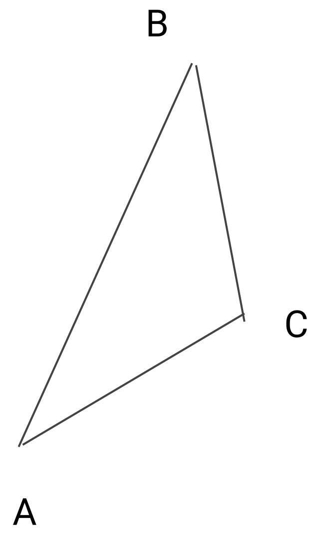
$$(\Sigma\, b(u) > m^2/n) \qquad \geq (1/n^2)\ n(m^2/n^2)^3$$

$$= m^6/n^7$$

*Mulholland-Smith (1959)/
Atkinson-Watterson-Moran (1959)/
Blakley-Roy (1965) inequality

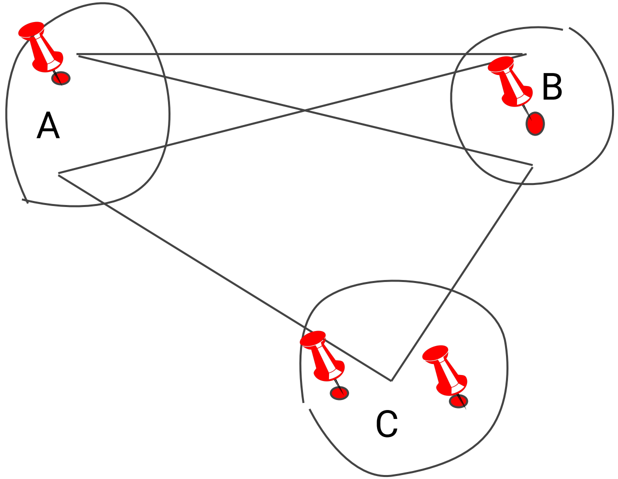
Five edges (similar triangles)

T1	A1	B1	C1
T2	A2	B2	C2
T3	A3	B3	C3
T4	A4	B4	C4
T5	A5	B5	C5



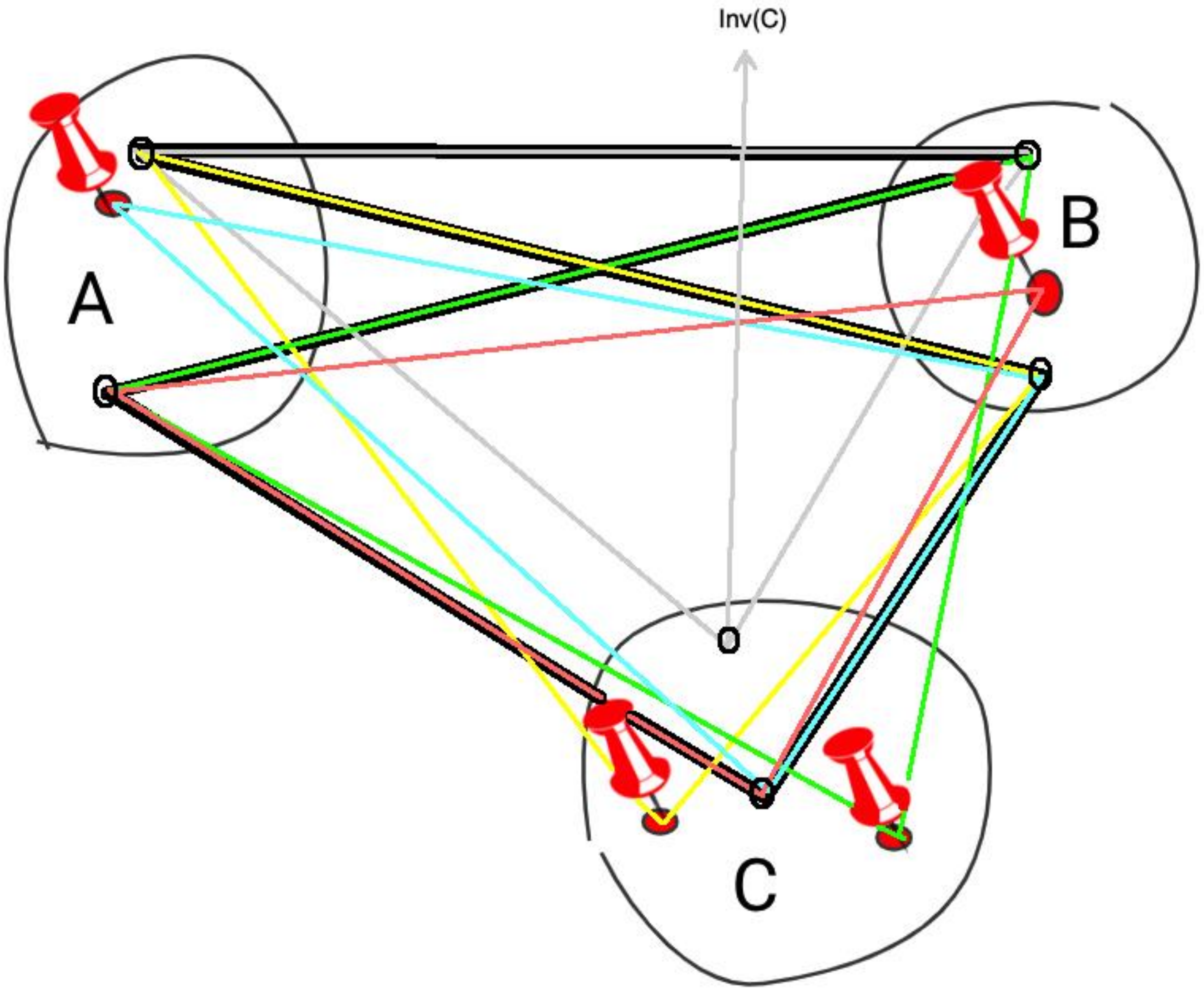
Five edges forming a pentagon

T1	A1	B1	C1
T2	A2	B2	C2
T3	A3	B3	C3
T4	A4	B4	C4
T5	A5	B5	C5



A1=A5, A2=A3, B1=B2, C3=C4, B4=B5

Fixed vertices: A4, B3, C1, C2


Inv(C)
A
B
C

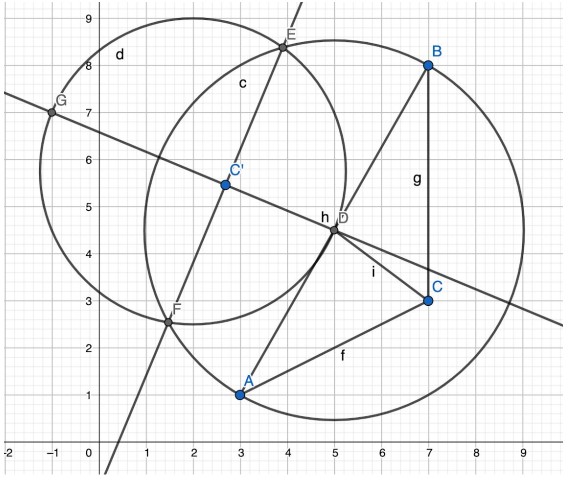
Pentagons with four pinned vertices

We want to show that the inverse of *C5* in *T5* is determined by the four fixed vertices. Since the triangles are similar, we can express *CI* as

$CI=(AI+BI)/2+z(AI-BI)/2$, where $z=re^{i\Theta}$, and

$inv(C5)=(A5+B5)/2+(A5-B5)/2z,$

for $I=1,2,3,4,5.$



Inv(C5) is determined by the four pinned vertices

The triangles are similar, so vertex *C* can be expressed by *A*, *B*, and a constant parameter.

$CI=(AI+BI)/2+z(AI-BI)/2$, where $z=re^{i\Theta}$, and

$\text{inv}(C5)=(A5+B5)/2+(A5-B5)/2z=(A5+B4)/2+(A5-B4)/2z$

After some calculations … we will see that

$(A3+B4)/2+(A3-B4)/2z=(A4+B3)/2+(A4-B3)/2z=*$

$*+C1-C2=inv(C5)$

$$\frac{A_3 + B_3}{2} + \frac{z(A_3 - B_3)}{2} = \frac{A_4 + B_4}{2} + \frac{z(A_4 - B_4)}{2},$$

$$A_3 \left(\frac{1+z}{2}\right) + B_3 \left(\frac{1-z}{2}\right) = A_4 \left(\frac{1+z}{2}\right) + B_4 \left(\frac{1-z}{2}\right),$$

$$A_3 \left(\frac{1+z}{2}\right) - B_4 \left(\frac{1-z}{2}\right) = A_4 \left(\frac{1+z}{2}\right) - B_3 \left(\frac{1-z}{2}\right),$$

$$A_3 \left(\frac{\frac{1}{z}+1}{2}\right) - B_4 \left(\frac{\frac{1}{z}-1}{2}\right) = A_4 \left(\frac{\frac{1}{z}+1}{2}\right) - B_3 \left(\frac{\frac{1}{z}-1}{2}\right),$$

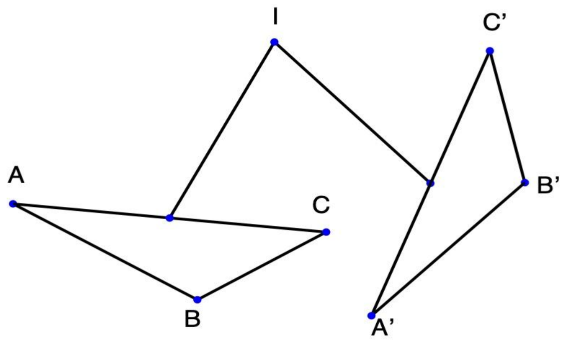
$$\frac{A_3 + B_4}{2} + \frac{A_3 - B_4}{2z} = \frac{A_4 + B_3}{2} + \frac{A_4 - B_3}{2z}.$$

$$
\begin{aligned}
C_1 - C_2 &= \frac{A_1 + B_1}{2} + \frac{z(A_1 - B_1)}{2} - \left(\frac{A_2 + B_2}{2} + \frac{z(A_2 - B_2)}{2} \right) \\
&= \frac{A_1}{2} - \frac{A_2}{2} + \frac{A_1}{2z} - \frac{A_2}{2z} \\
&= \frac{A_5}{2} - \frac{A_3}{2} + \frac{A_5}{2z} - \frac{A_3}{2z}.
\end{aligned}
$$

$$
Inv(C_5) = \frac{A_4 + B_3}{2} + \frac{A_4 - B_3}{2z} + C_1 + C_2.
$$

The Key Lemma we wanted to prove

Lemma: There is a universal constant, *c>0,* such that the following holds. Let us suppose that *n* points in the plane span at least $n^{11/6}$ triangles, similar to a given triangle *T*. Then for any selection of $n^{11/6}$ triangles similar to *T*, there will be at least two vertex disjoint triangles sharing the same inversion point.

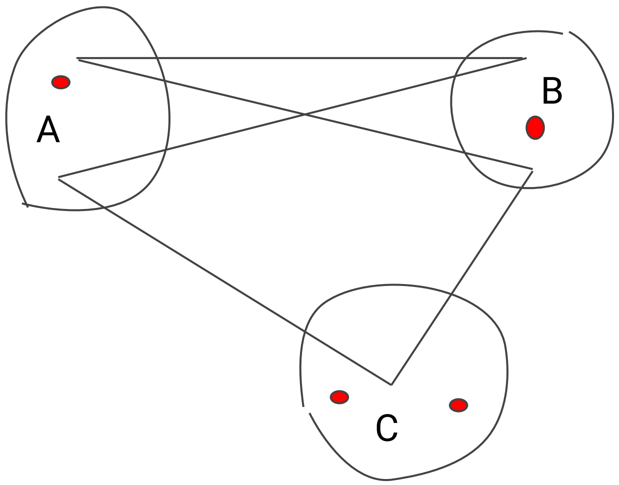


Proof

If on n points we have at least $m>Cn^{11/6}$ triangles, similar to a given triangle T, then the number of Berge pentagons is $>n^4$, so there are at least two pentagons with the same *A4, B3, C1, C2* vertices. Then in both pentagons *Inv(C5)* is the same point.

Five edges forming a pentagon

T1	A1	B1	C1
T2	A2	B2	C2
T3	A3	B3	C3
T4	A4	B4	C4
T5	A5	B5	C5



A1=A5, A2=A3, B1=B2, C3=C4, B4=B5

Fixed vertices: A4, B3, C1, C2

Elekes and Erdős

Elekes and Erdős proved that for any triangle, T, there are n-element planar point sets with $O(n^2)$ triangles similar to T.

It was proved shortly after that if the number of equilateral triangles is at least $(1/6+\varepsilon)n^2$ then the pointset should contain large parts of a triangular lattice.

On the other hand, no lattice is guaranteed by cn^2 similar copies if $c<1/6$.

Corollaries

There are some interesting corollaries of the Key Lemma and pentagon counting.

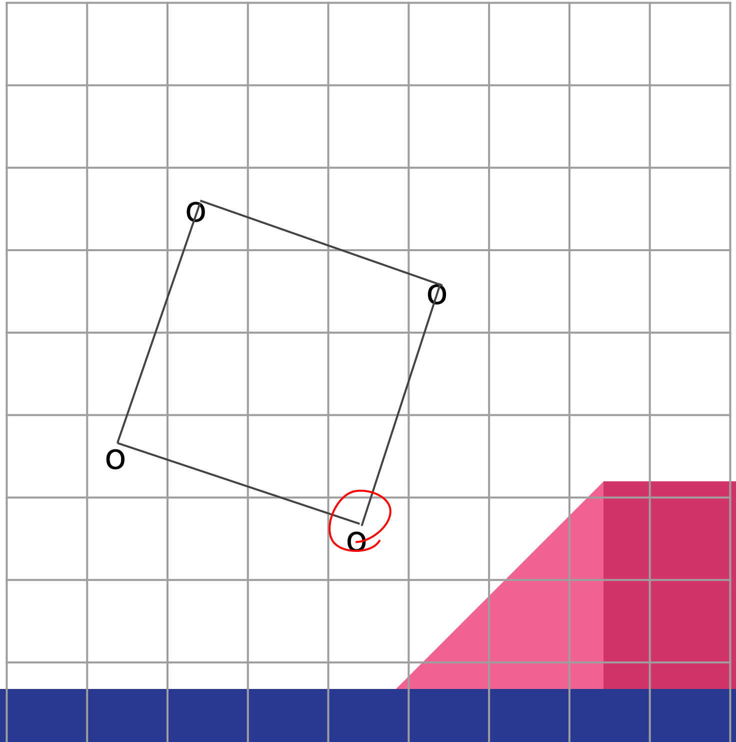
1. For every $k>3$ and $c>0$ there is a constant, C, such that if n points span at least cn^2 similar triangles, then it is a subset of a pointset of size Cn which spans at least cn^2 similar k-gons.
2. The next question was *Problem 25.* in the 12th Gremo Workshop on Open Problems 2014, Val Sinestra (GR), Switzerland, Jun 30 - Jul 4, 2014.

Problem 25.

Given an [n]x[n] integer grid. We are choosing grid points as long as there is no square among the selected points.

<u>Claim</u>: At least $n^{12/11}$ points

should be selected before stuck.



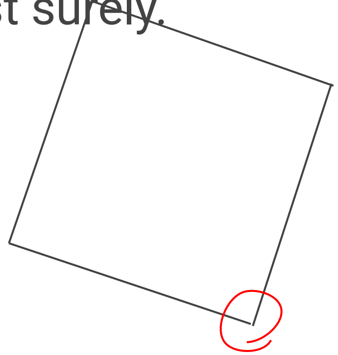
Proof of the Claim

When we can’t add a new grid point, then every empty place is “covered” by a right isosceles triangle, i.e. every empty grid point is the inversion point of the right angle vertex of a right isosceles triangle. Let us suppose that the number of selected points is m. By considering a subset of the triangles **we can assume that every empty grid point is the inversion point of exactly one triangle**. The number of empty grid points is about n^2 (or we are done). But then, from our Key Lemma we know that the number of the right isosceles triangles, which is about n^2, is at most $m^{11/6}$. So, we have $m>n^{12/11}$ as we wanted.

Problem 25.

Given an [n]x[n] integer grid. We are choosing grid points at random as long we can such that there is no square among the selected points.

Claim (Jozsi Balogh, Igor Araujo): The random greedy process stops after selecting not more than $n^{191/86}$ points almost surely.



(Based on Bennett-Bohman)

Larger polygons

Using more advanced calculations one can prove a better bound on quadrilaterals.

Lemma: There is a universal constant, *c>0,* such that the following holds. Let us suppose that *n* points in the plane span at least $n^{7/4}$ quadrilaterals, similar to a given quadrilateral Q. Then for any selection of $n^{7/4}$ triangles similar to *Q*, there will be at least two vertex disjoint triangles sharing the same "*inversion*" point.

Are there better bounds for pentagons?

Open Problems

We gave a bound on the number of pentagons in 3-uniform, linear hypergraphs. Is it sharp? In a random linear hypergraph we have more pentagons in the range we used.

Is there a way to use structure given by the geometric settings, instead of working with arbitrary linear hypergraphs?

What is the answer to Problem 25? What is the smallest maximal square-free subset of $[n]x[n]$? What is the largest? When will the random greedy random process stop? (Between $n^{4/3}$ and $n^{191/86}$ (Igor Araujo))