On the Zarankiewicz problem for graphs with bounded VC-dimension

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Joint work with Cosmin Pohoata (Yale)

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This bound is known to be tight for $t = 2$ and $t = 3$.
However, for $t = 4$, the best known lower bound is

$$\mathrm{ex}(n, K_{4,4}) \geq \mathrm{ex}(n, K_{3,3}) = \Omega(n^{5/3}).$$

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Theorem (Alon–Kollár–Rónyai–Szabó '99)

For $s \geq (t-1)! + 1$,

$$\mathrm{ex}(n, K_{t,s}) = \Theta(n^{2-1/t}).$$

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Define the VC-dimension of G to be the VC-dimension of this set system.

Shatter function

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For a set system $\mathcal{F}$ on ground set V, the shatter function is defined as

$$\pi_{\mathcal{F}}(z) = \max_{S \subset V : |S| = z} |\{A \cap S : A \in \mathcal{F}\}|.$$

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That is, $\pi_{\mathcal{F}}(z) \leq cz^d$ for some $c = c(d)$.

The result of Fox, Pach, Sheffer, Suk and Zahl

Theorem (Fox–Pach–Sheffer–Suk–Zahl '17)

Let G be a bipartite graph with parts A and B such that $|A| = m$ and $|B| = n$ such that the set system $\mathcal{F} = \{N(b) : b \in B\}$ has $\pi_{\mathcal{F}}(z) \leq cz^d$. If G is $K_{t,t}$-free, then

$$e(G) \leq c_1(mn^{1-1/d} + n),$$

where $c_1 = c_1(c, d, t)$.

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Corollary

Let G be a $K_{t,t}$-free bipartite graph on $n + n$ vertices which has VC-dimension at most d. Then

$$e(G) \leq cn^{2-1/d},$$

where $c = c(d, t)$.

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- (It is because for any $T \subset S$ of size d there are less than $d!$ common neighbours of T.)
- Hence,

$$\pi_{\mathcal{F}}(z) \leq d! \cdot \binom{z}{d} + \sum_{i=0}^{d-1} \binom{z}{i}.$$

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However, we can improve the upper bound for all $d \geq 3$.

Theorem (J.–Pohoata '20+)

Let $d \geq 3$ and t be fixed positive integers. Let G be a $K_{t,t}$-free bipartite graph on $n + n$ vertices which has VC-dimension at most d. Then

$$e(G) = o(n^{2-1/d}).$$

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- $F(d)$ has parts X and Y where $|X| = d+1$ and $|Y| = 2^{d+1}$ and for every $Z \subset X$ there is a vertex $y_Z \in Y$ such that the neighbourhood of y_Z in $F(d)$ is precisely Z.

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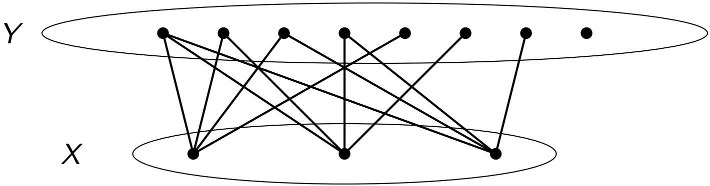

Figure: The graph $F(2)$

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Our proof is inspired by the proof of this result.

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We want to show that if G has at least $cn^{2-1/d}$ edges, then either A has a subset of size $d+1$ which is shattered, or G has $K_{t,t}$ as a subgraph.

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Lemma

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1. *there exists a set $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $|N(D)| \geq r$ or*
2. *there exist $\Theta(|N(x)|^r)$ sets $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $N(D) \setminus \{x\} \neq \emptyset$.*

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On the next slide we prove that if 1) holds, then a randomly chosen subset of R of size $d+1$ is shattered with positive probability, unless G contains $K_{t,t}$.

If we have a "rich" set R

- Assume that case 1) holds from the previous slide, i.e. that for some large (compared to d and t) constant r there exist $x \in B$ and $R \subset N(x)$ of size r such that for every $D \subset R$ of size d we have $N(D) \geq r$.

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- Hence, if we choose $S \subset R$ of size $d+1$ randomly, then with positive probability we will have that $|N(z) \cap N(D)| < \frac{1}{d+1}|N(D)|$ holds for every $D \subset S$ of size at most d and every $z \in S \setminus D$.

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- So, for each $D \subset S$ of size at most d, we may choose some $b_D \in B$ such that $N(b_D) \cap S = D$.
- Using $S \subset N(x)$, this shows that S is shattered.

If we have many "decent" candidates

- Assume now that case 2) holds, i.e. for some large r there exist $\Theta(|N(x)|^r)$ sets $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $N(D) \setminus \{x\} \neq \emptyset$.

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  - the neighbourhoods of the d-subsets of R are not sufficiently disjoint that we can assign a different neighbour for each d-subset.
- In either scenario, we can argue that the number of such bad sets R is small.
- For example, assume that $N(D) = N(D') = \{x, y\}$ for some distinct subsets D and D' of R of size d.

If we have many "decent" candidates

- Assume now that case 2) holds, i.e. for some large r there exist $\Theta(|N(x)|^r)$ sets $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $N(D) \setminus \{x\} \neq \emptyset$.
- Roughly speaking, two possible problems can occur:
  - either some subset of R of size at most $d-1$ has small common neighbourhood, or
  - the neighbourhoods of the d-subsets of R are not sufficiently disjoint that we can assign a different neighbour for each d-subset.
- In either scenario, we can argue that the number of such bad sets R is small.
- For example, assume that $N(D) = N(D') = \{x, y\}$ for some distinct subsets D and D' of R of size d.
- Then the number of ways to choose $D \cup D'$ is $o(|N(x)|^{|D \cup D'|})$ since once we have chosen D, the vertex y is determined, and every member of $D' \setminus D$ must be a neighbour of y.

The proof of the lemma

Lemma

Let G be a bipartite graph with parts A and B and with minimum degree satisfying $\delta(G) \geq cn^{1-1/d}$ for some constant $c > 0$, and where $|A|, |B| \leq n$. Let r be a constant positive integer and let $x \in B$. Then one of the following two statements must be true:

1. *there exists a set $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $|N(D)| \geq r$ or*
2. *there exist $\Theta(|N(x)|^r)$ sets $R \subset N(x)$ of size r such that for every $D \subset R$ of size d, we have $N(D) \setminus \{x\} \neq \emptyset$.*

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Define a d-uniform hypergraph $\mathcal{H}$ on vertex set $N(x)$ such that a set $D \subset N(x)$ of size d is a hyperedge of $\mathcal{H}$ if and only if $N(D) \setminus \{x\} \neq \emptyset$.

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Now condition 2) is saying that $\mathcal{H}$ contains $\Theta(|N(x)|^r)$ copies of $K_r^{(d)}$.

The proof of the lemma

- By the hypergraph removal lemma, it suffices to prove that in order to destroy all copies of $K_r^{(d)}$ in $\mathcal{H}$, one needs to remove $\Theta(|N(x)|^d)$ hyperedges from $\mathcal{H}$.

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- Colour a d-set $D \subset N(x)$ green if $N(D) = \{x\}$, blue if $1 < |N(D)| < r$ and red if $|N(D)| \geq r$.

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- However, if ℓ is sufficiently large, by Ramsey's theorem we know that for every set $L \subset N(x) \cap N(y)$ of size ℓ, there exists a subset $R \subset L$ of size r such that all d-sets in R have the same colour.

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- So we can assume that they are all blue.

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- So we can assume that every ℓ-set in every $N(x) \cap N(y)$ contains a $K_r^{(d)}$ in which all hyperedges are blue.

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- Hence, to destroy every $K_r^{(d)}$, we need to delete at least one blue edge from each ℓ-set in each $N(x) \cap N(y)$.

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$$\frac{1}{r} \sum_{y \in B \setminus \{x\}} \frac{1}{\binom{\ell}{d}} \binom{d(x,y)}{d}$$

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- Since the minimum degree of the graph is at least $cn^{1-1/d}$, it is easy to see that the sum is $\Omega(|N(x)|^d)$.

Open problems

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The result of Fox, Pach, Sheffer, Suk and Zahl had many geometric applications. Does our VC-dimension version have geometric applications?

Thank you for your attention!