

Functional Löwner
ellipsoids
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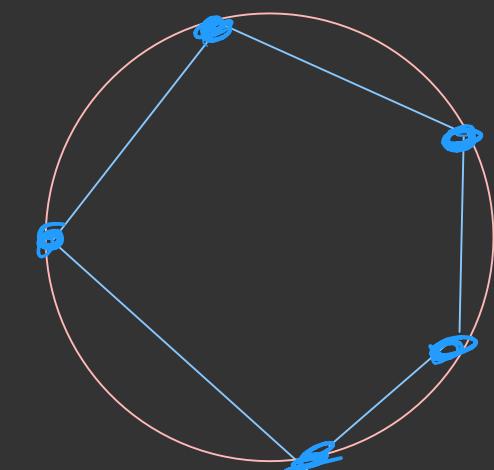
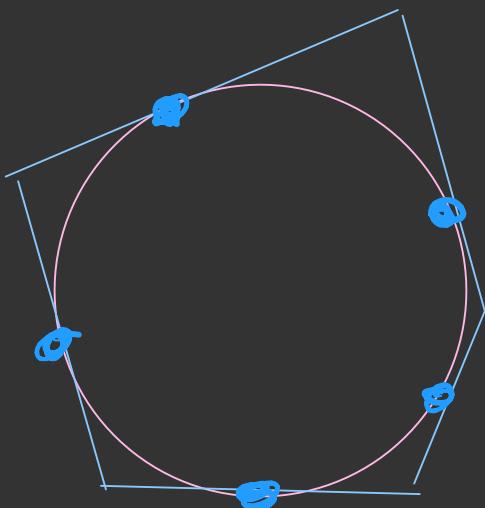
I. Prologue

John and Löwner ellipsoids

Let $K \subset \mathbb{R}^d$ be a convex body.

John ellipsoid J_K
maximal volume,
contained in K

Löwner ellipsoid L_K
minimal volume
containing K



Many Properties!

- ① They exist for any convex body and are unique!
- ② John's condition (due to John and Ball)
- $B^d = J_K$ iff $\exists (u_i)_1^m \subset \partial K \cap \partial B^d$ and $(c_i)_1^m \subset \mathbb{R}_+$
- s.t. $\sum c_i u_i \otimes u_i = Id$,
 $\sum c_i u_i = 0$
- ③ Dual via Minkowski's polarity:
- $(J_K)^\circ = L_K^\circ$

$$\textcircled{4} \quad \text{vr}(K) = \left(\frac{\text{vol } K}{\text{vol } J_K} \right)^{1/d}, \quad \text{ovr}(K) = \left(\frac{\text{vol } L_K}{\text{vol } K} \right)^{1/d}$$

L. Ball $\left\{ \begin{array}{l} 1 \leq \text{vr}(K) \leq \text{vr}(\Delta^d) \\ \leq \text{vr}(\square^d) \end{array} \right\} \sim \sqrt{d}$

F. Barthe $\left\{ \begin{array}{l} 1 \leq \text{ovr}(K) \leq \text{ovr}(\Delta^d) \\ \leq \text{ovr}(\square^d) \end{array} \right\}$

The log-concave stuff.

Def. $f: \mathbb{R}^d \rightarrow [0; +\infty)$ is log-concave, if

$$f = e^{-\psi},$$

where $\psi: \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex.

Examples

① $\chi_K(x) = \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$

② $(2\pi)^{-d/2} e^{-|x|^2} *$

③ The marginal on \mathbb{R}^d of the uniform measure
on a convex body on $\mathbb{R}^{d+1} **$

* and probably any other probability distribution you
may now recall.

** in fact, any log-concave function is a limit of such
marginals

The question is...

How do we define
Löwner ellipsoids
for log-concave functions?

II. The First Approach: Li, Schütt, Werner, 2019

- a log-concave $f: \mathbb{R}^d \rightarrow [0, +\infty)$
- consider log-concave $e^{-|x|}$.
- among affine images $\alpha e^{-|\mathcal{A}(x-a)|} \geq f$

find one of minimal integral

Theorem (LSW'19)

Let $f: \mathbb{R}^d \rightarrow [0, +\infty)$ be log-concave.

A solution to the problem

$$\int \alpha e^{-|\mathcal{A}(x-a)|} \rightarrow \min \text{ subject to } \alpha e^{-|\mathcal{A}(x-a)|} \leq f$$

exists and is unique.

The solution is called the Löwner function of f

III. Why not ask for more? Ivanov, T., 2020.

- a log-concave $f: \mathbb{R}^d \rightarrow [0; +\infty)$
- consider a radial log-concave $e^{-\psi(|x|)}$
- among affine images $\alpha e^{-\psi(|A(x-\alpha)|)} \geq f$,
find one of minimal integral

features:

- the super-level sets of $\alpha e^{-\psi(|A(x-\alpha)|)}$ are concentric ellipsoids
- $\psi(t)=t$ is the LSW-case
- $\psi(t)=t^2$ — the case of "covering" the function f by a Gaussian of the minimal integral

Theorem (Ivanov, T. '20)

$f: \mathbb{R}^d \rightarrow [0, +\infty)$ be log-concave and

$\psi: [0, +\infty) \rightarrow \mathbb{R}$ be a strictly increasing convex function

Then a solution $L_f[\psi]$ to the problem

$$\int \alpha e^{-\psi(\|A(x-a)\|)} \rightarrow \min \text{ subj. to } \alpha e^{-\psi(\|A(x-a)\|)} \geq f$$

exists* and is unique. Moreover,

$$\|L_f[\psi]\| \leq e^d \|f\|.$$

$L_f[\psi]$ — Löwner ψ -function of f .

Interesting particular case (it really is)

Fix $\delta > 0$. Put $\gamma_\delta = \frac{\delta}{2} \left[\sqrt{1 + 4\left(\frac{\epsilon}{\delta}\right)^2} - \ln \left(\frac{1 + 4\left(\frac{\epsilon}{\delta}\right)^2}{2} \right) - 1 \right]$

Then the problem

$$\int \alpha e^{-\gamma_\delta(|A(x-a)|)} \rightarrow \min \text{ subj. to } \alpha e^{-\gamma_\delta(|A(x-a)|)} \leq f$$

is dual to

$$\int \alpha (1 - A^{-\delta} |x-a|^2)^{s/2} \rightarrow \max \text{ subj. to } \alpha (1 - A^{-\delta} |x-a|^2)^{s/2} \leq f.$$

(studied by Ivanov and Naszodi '20)

- recall that $O.v.rat(\mathcal{L}) = \left(\frac{\text{vol } \mathcal{L}}{\text{vol } \mathcal{L}_k} \right)^{1/d}$
- $\mathcal{L}_f[\text{id}] = \mathcal{L}_f = L_{IW}$ Lower function =
 $= \alpha e^{-\|A(x-a)\|} f$ of minimal integral

$$O.i.rat(f) = \left(\frac{\int \mathcal{L}_f}{\int f} \right)^{1/d}$$

Theorem (Ivanov, T. '20)

Let $f: \mathbb{R}^d \rightarrow [0; +\infty)$ be log-concave. Then

$$O.i.rat(f) \leq O(d)$$

Sketches of the proofs.

Uniqueness

interpolation + Minkowski
determinant inequality.

↓
extremizers are
translates

↓
"sausage" trick

↓
uniqueness

Existence

boundedness of
the parameters

↓
compactness

↓
extrema are
attained

Outer integral ratio

Theorem (Ivanov, T.'20)

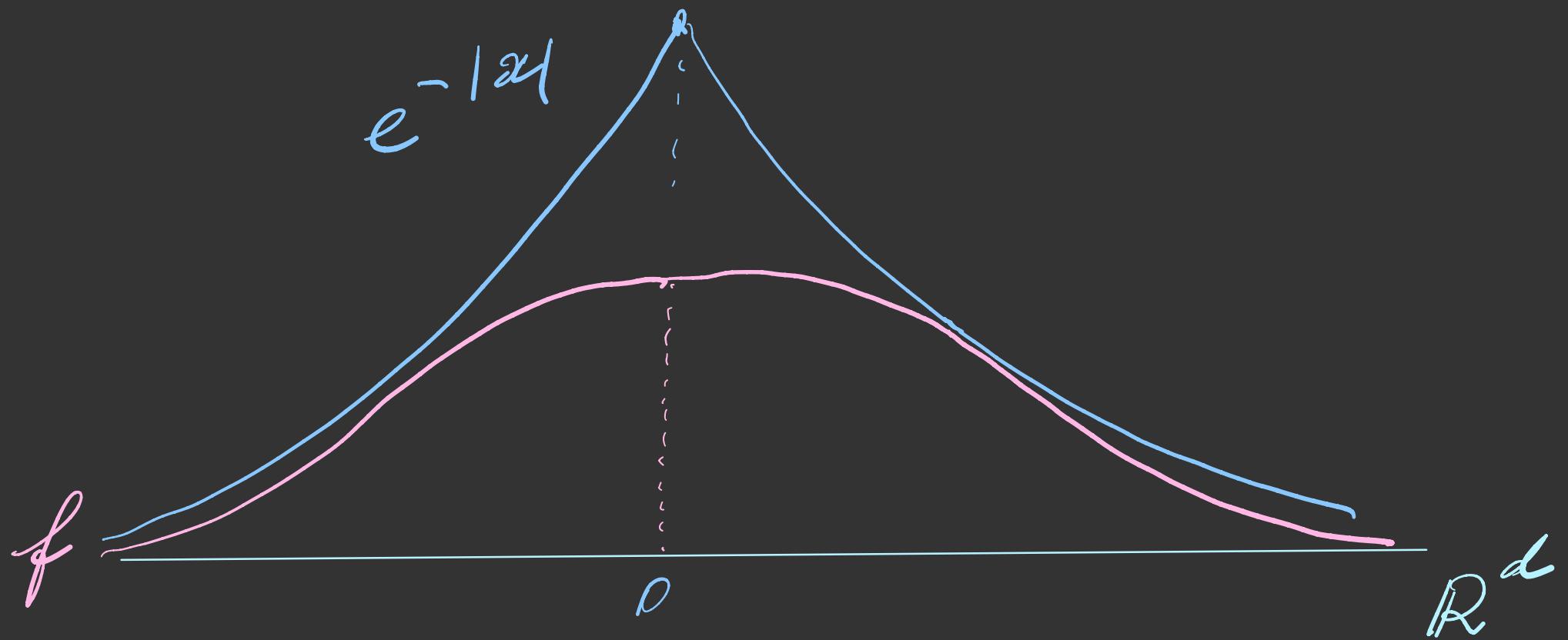
$$\text{O.i.rat}(f) = \left(\frac{\int_L f}{\int f} \right)^{1/d} \leq \Theta \sqrt{d}$$

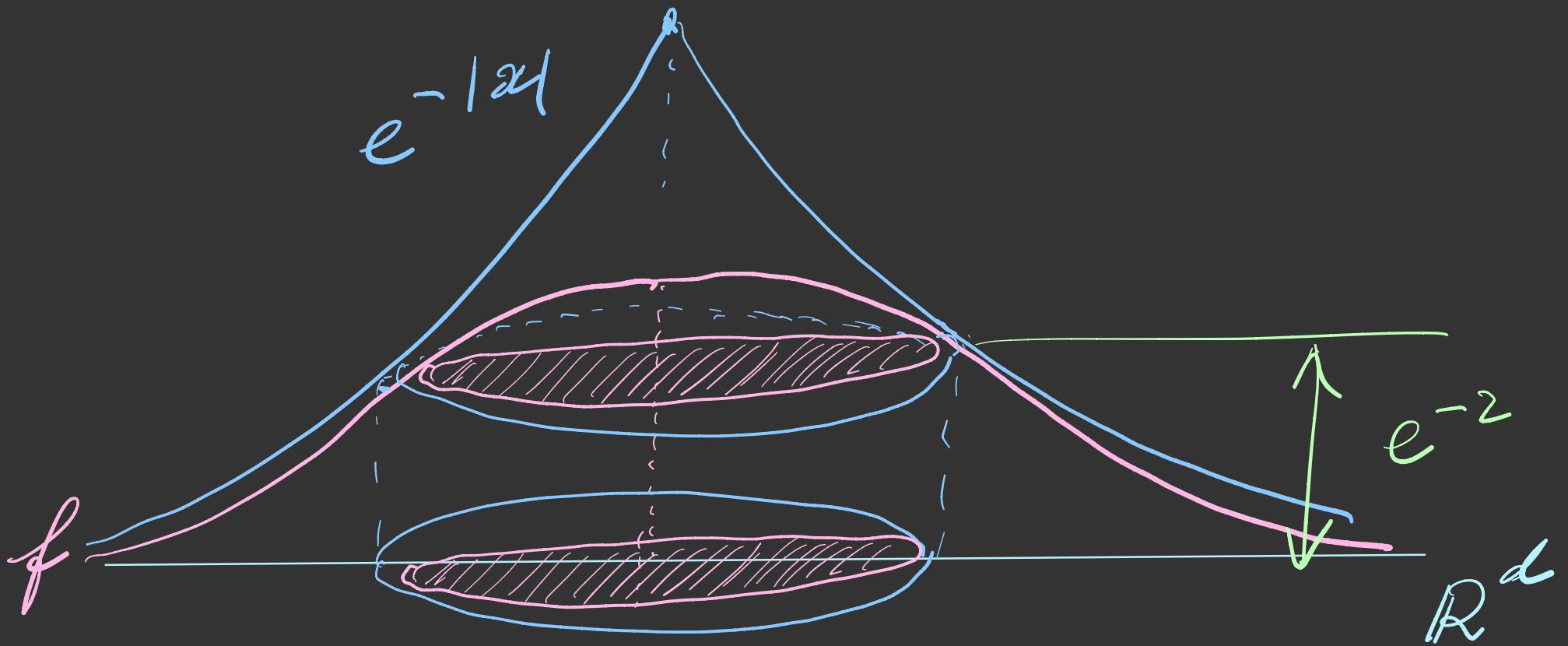
Now:

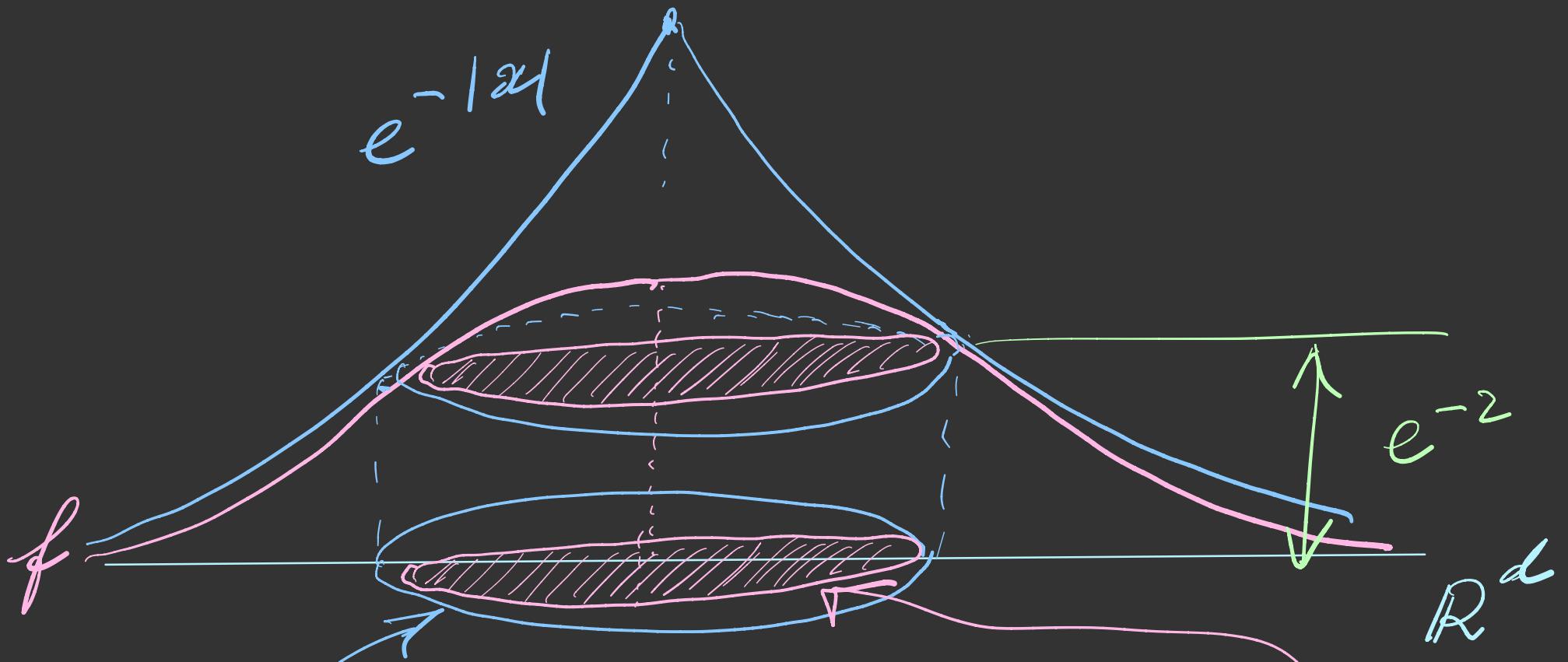
- outer integral ratio is affine invariant
- the trick is to find a convenient position
- assume that $\|f\| = f(0) = e^{-d}$ and that $f \leq e^{-|Ax|}$ is the minimal integral function of height 1.
- WLOG $A = Id$

$$\text{O.i.rat}(f) = \left(\frac{\int \mathcal{L}_f}{\int f} \right)^{1/d} \leq \left(\frac{\int e^{-|x|}}{\int f} \right)^{1/d}$$

now: a lower bound on $\int f$!







$$2B^d \supset \{x \in \mathbb{R}^d : f(x) \geq e^{-2}\}$$

$$\mathcal{K}_f = \bigcup_{r \geq d} \frac{1}{r} \{x \in \mathbb{R}^d : f(x) \geq e^{-2}\}$$

$$\mathcal{K}_f = \bigcup_{x \in \mathbb{R}^d} \frac{1}{2} \{x \in \mathbb{R}^d : f(x) \leq e^{-2}\}$$

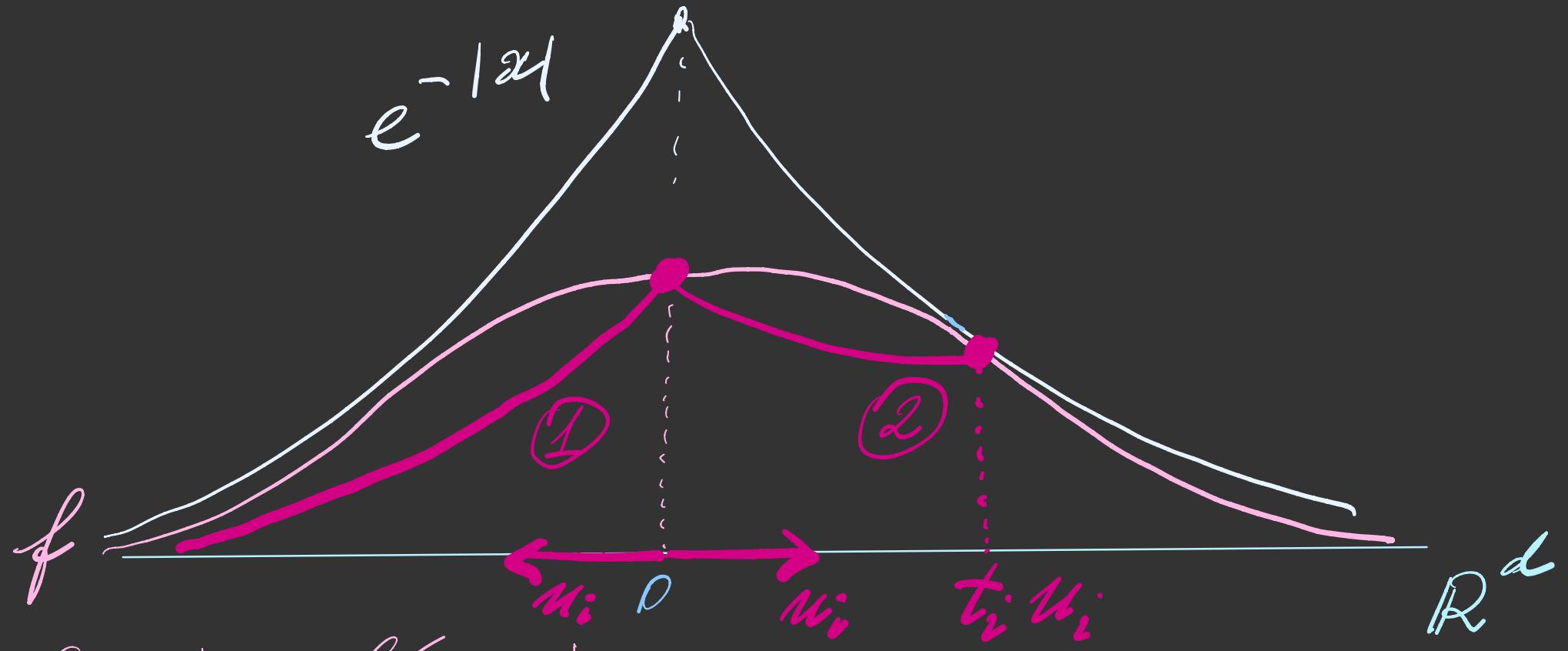
a new functional mapping!

Observation: $\mathcal{K}_f \subset B^d$,

and B^d is the minimal volume ellipsoid centered in the origin containing \mathcal{K}_f .

Then, there exist $(u_i)_{i=1}^m \subset \partial \mathcal{K}_f \cap \partial B^d$ and $(c_i) \subset \mathbb{R}_+$ such that

$$\sum_{i=1}^m c_i u_i \otimes u_i = Id \text{ and } \sum_{i=1}^m c_i = d.$$



$$\textcircled{1} \quad f_i(t) \geq e^{-d} e^{-t}$$

$$\textcircled{2} \quad f_i(t) \leq e^{-d} e^{-t} \left(1 - \frac{d}{t_i}\right) \text{ for } t \in [0, t_i]$$

Theorem (Barthe '98) [Reverse Brascamp-Lieb ineq.]

Let $(u_i)_{1}^m \subset \mathbb{R}^d$ and $(c_i)_{1}^m \subset \mathbb{R}_+$ be s.t.

$$\sum c_i u_i \otimes u_i = \text{Id}$$

and $(q_i)_{1}^m \subset L_1(\mathbb{R})$, $q_i \neq 0$. Then

$$\int \sup \left\{ \prod_{i=1}^m [q_i(\theta_i)]^{c_i} \mid \theta = \sum c_i \theta_i u_i \right\} d\theta \geq \prod_{i=1}^m \left(\int q_i \right)^{c_i}$$

log-concavity of f yields

$$f(x) \geq \sup \left\{ \prod_{i=1}^m f_i(d \cdot \theta_i)^{c_i/d} \mid x = \sum c_i \theta_i u_i \right\}$$

This and the reverse Brascamp-Lieb ineq. implies

$$\int f \geq e^{-d}$$

*Thank you
for your
attention!*