

Crossings between non-homotopic edges

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Graph G :

Crossing number $\text{cr}(G)$: min. number of edge crossings
over all drawings of G
(on the plane)

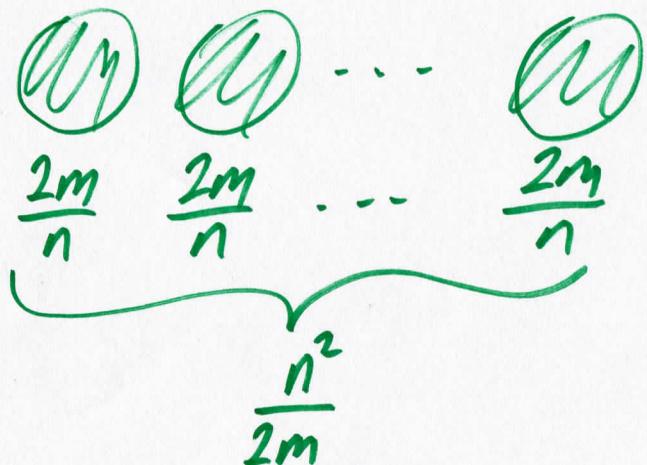
Crossing Lemma (Ajtai-Chvátal-Newborn-Szemerédi 82,
Leighton 83)

G simple graph, n vertices, $m \geq 4n$ edges \Rightarrow

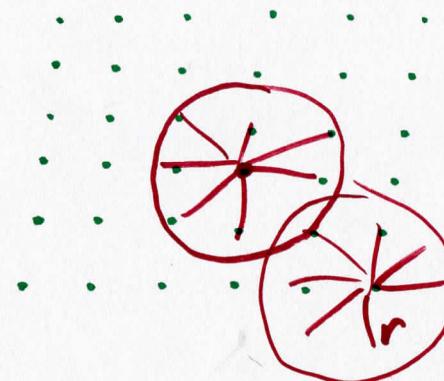
$$\Rightarrow \text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}$$

Ackerman 15: $m \geq 7n \Rightarrow \text{cr}(G) \geq \frac{1}{29} \cdot \frac{m^3}{n^2}$

Asymptotically tight:



$$\text{cr}(G) \approx c \cdot \frac{m^3}{n^2}$$



Vertices: grid.

Edges: iff $|uv| < r$

$$\text{cr}(G) \approx \frac{8}{9\pi^2} \cdot \frac{m^3}{n^2}$$

Pach-T 97

Czabarka - Singgih - Sze'kely - Wang 20

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Does the Crossing Lemma hold for multigraphs?

NO:



or



Székely '97: G multigraph, n vertices, m edges
max edge multiplicity: k

$$m \geq 5kn \Rightarrow \text{cr}(G) \geq c \cdot \frac{m^3}{k \cdot n^2}$$

Agoston-PálVölgyi: Holds with the same c as the Cr. Lemma.

When can we replace $\frac{m^3}{k \cdot n^2}$ by $\frac{m^3}{n^2}$?

Drawing of multigraph: topological multigraph.

1. separated:  A vertex between any two parallel edges.

2. locally starlike:  Adjacent edges do not cross.

Pach-T 18: G separated and locally starlike top. multigraph,

$$m \geq 4n \Rightarrow \text{cr}(G) \geq c \cdot \frac{m^3}{n^2} \quad (\text{asymp. tight})$$

$$\text{max number of edges: } c \cdot n^2$$

Kaufmann - Pach - T - Veckert 19:

G separated top. multigraph,

$$m \geq 4n \Rightarrow \text{cr}(G) \geq c \cdot \frac{m^{2,5}}{n^{1,5}} \quad (\text{asymp. tight})$$

$$\text{max number of edges: } c \cdot n^3$$

Weaker conditions?

still have to rule out



Non-homotopic topological multigraph:

- No two parallel edges are homotopic
(vertices \leftrightarrow holes)
- No trivial loops

Self-crossing allowed.

$cr(G)$: number of crossings

Non-homotopic top. multigraph, n vertices ($n \geq 2$)
max number of edges? $\infty!$



$\text{cr}(n,m)$: min cr(G) of non-homotopic top multigraph G of n and m edges.

Pach-Tardos-T : $m \geq 4n$:

$$\frac{1}{24} \frac{m^2}{n} \leq \text{cr}(n,m) \leq 30 \frac{m^2}{n} \log^2 \frac{m}{n}$$

PTT For any fixed $n \geq 2$

$$\lim_{m \rightarrow \infty} \frac{\text{cr}(n,m)}{m^2} = \infty$$

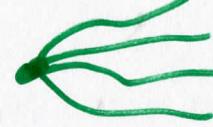
We proved $\text{cr}(n,m) \geq c \cdot m^2 \cdot \frac{\log^{1/6n} m}{n^7}$

Combining with Juvan-Malnič-Mohar 96: $\text{cr}(n,m) \geq c \cdot m^2 \cdot \frac{\log m}{n^8 \log \log m}$

Ideas of proofs.

Pach T 13: G separated, locally starlike topological multigraph,

$$m \geq 4n \Rightarrow \text{cr}(G) \geq c \cdot \frac{m^3}{n^2}$$

separated:  locally starlike: 

Let's try the standard (probabilistic) proof of the
Crossing lemma.

$$G \text{ simple graph } m \geq 4n \Rightarrow \text{cr}(G) \geq c \cdot \frac{m^3}{n^2}$$

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Crossing Lemma (G simple $m \geq 4n \Rightarrow \text{cr}(G) \geq C \cdot \frac{m^3}{n^2}$)

1. $\text{cr}(G) = 0 \Rightarrow G$ planar $\Rightarrow m \leq 3n - 6$

2. $\text{cr}(G) \geq m - 3n$. Proof: induction on m .

$m > 3n$, best drawing: there is crossing remove an edge from a crossing.

3. Random sampling. $m \geq 4n$, best drawing. $p = \frac{4n}{m} \leq 1$

Take each vertex with prob. $p \Rightarrow G'$

$$E(n') = pn \quad E(m') = p^2 \cdot m \quad E(\text{cr}(G')) \leq p^4 \cdot \text{cr}(G)$$

$$\text{cr}(G') \geq m' - 3n'$$

$$p^4 \cdot \text{cr}(G) \geq p^2 \cdot m - 3pn \quad p = \frac{4n}{m}$$

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}$$

Try the same for separated, locally starlike top. multigraphs.



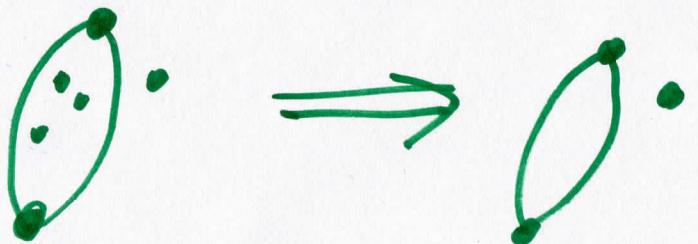
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1. $\text{cr}(G) = 0 \Rightarrow G$ planar $\Rightarrow m \leq 3n - 6$ (Euler formula) O.K.

2. $\text{cr}(G) \geq m - 3n$. induction on m O.K.

remove edge: still separated, loc. starlike

3. Random sampling: NO



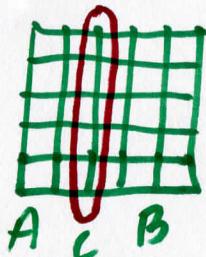
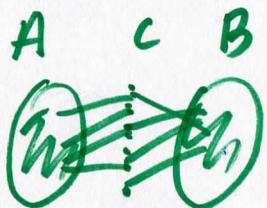
Need different method.

Lipton-Tarjan separator theorem:

G planar, n vertices \Rightarrow there is a partition

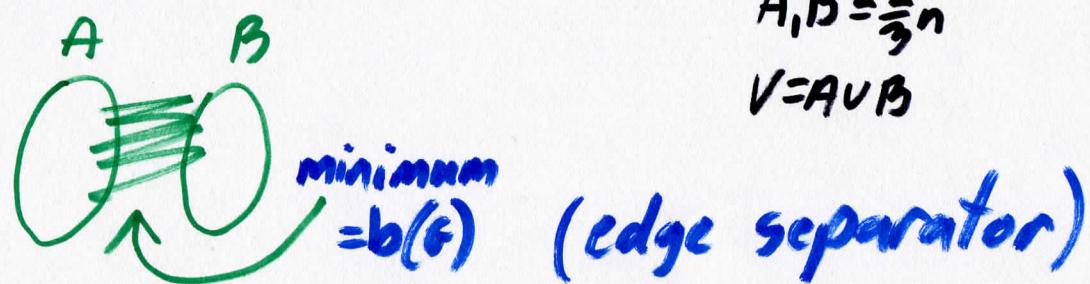
$$V(G) = A \cup B \cup C \quad |A|, |B| \leq \frac{2}{3}n \quad |C| \leq 4\sqrt{n} \quad \text{and}$$

no edge between A and B



C : vertex-separator.

Bisection width $b(G) = \min_{A, B \subseteq \frac{2}{3}n} E(A, B)$

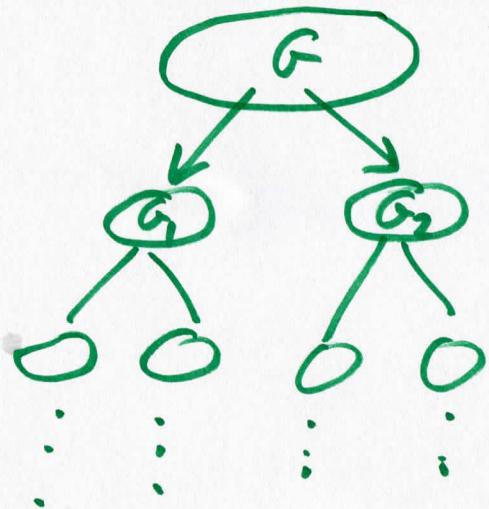


Pach-Shahrabi-Gregory, Alon-Seymour-Thomas, Barit-Hiller...

G : n vertices, degrees d_1, d_2, \dots, d_n

$$\Rightarrow b(G) \leq 10\sqrt{cr(G)} + 2\sum d_i^2$$

Crossing Lemma, complicated proof:



$$\text{Want: } \text{cr}(G) \geq c \cdot \frac{m^3}{n^2}$$

G_i : cut (remove $\sim \sqrt{\text{cr}(G_i)}$ edges)
 if $\text{cr}(G_i)$ is large, STOP
 if small → recursively

Finally: removed $o(m)$ edges. Typical component: $\approx \frac{m}{n}$ vertices
 $\approx \frac{m^2}{n^2}$ edges $\text{cr}(G_i) \approx \frac{m^4}{n^4}$ $\rightarrow \text{cr}(G) \geq c \cdot \frac{n^2}{m} \cdot \frac{m^4}{n^4} = c \cdot \frac{m^3}{n^2}$

+ Some modification: Works for separated, loc. starlike top. multigraphs!

Non-homotopic topological multigraphs

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$$T: m \geq 4n : \frac{1}{24} \frac{m^2}{n} \leq cr(n, m) \leq 30 \frac{m^2}{n} \log^2 \frac{m}{n}$$

Loose non-homotopic top. multigraph: distinct edges do not cross. Self-crossing O.K.

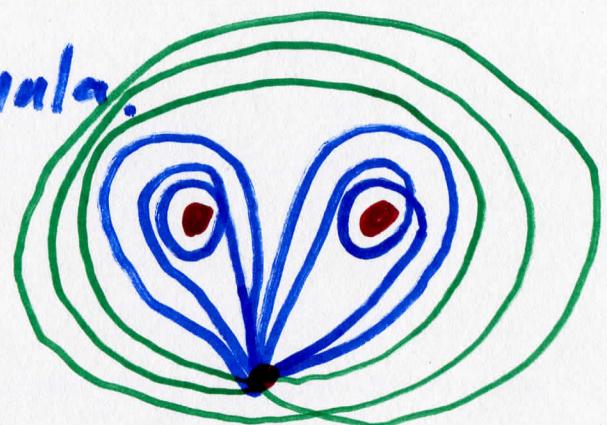
Lemma 1: Max number of edges of n -vertex loose non-homot. top. multigraph on sphere: $3n-6$ for $n \geq 3$
 1 for $n=2$

Lemma 2: In the plane: $3n-3$ for $n \geq 1$.

$n=3$ $m=6$

Proof: If no self-crossing: Euler's formula.

In general: tricky induction.



G : n vertices, m edges, non-homot. top. multigraph.

D : non-crossing graph of edges.

m vertices, no complete subgraph of $3n-2$ vertices

\Rightarrow Turán's theorem:

$$|E(D)| \leq \frac{m^2}{2} \left(1 - \frac{1}{3n-3}\right)$$

$$\Rightarrow cr(G) \geq \binom{m}{2} - |E(D)| \geq \frac{1}{24} \cdot \frac{m^2}{n}$$

$$cr(n, m) \leq 30 \frac{m^2}{n} \log^2 \frac{m}{n}$$

Let $n=3$.

x -loops \leftrightarrow words of $a, b, a^{-1} b^{-1}$

Take all $a-b$ words of length $\log m \Rightarrow m$ loops.

$\underbrace{aab\dots bb}_{\log m}$

Each of them: polygon of $3 \log m$ segments.

$\Rightarrow \leq 9 \log^2 m$ pairwise crossings.

$$cr \leq c \cdot m^2 \log^2 m.$$

$n > 3$:

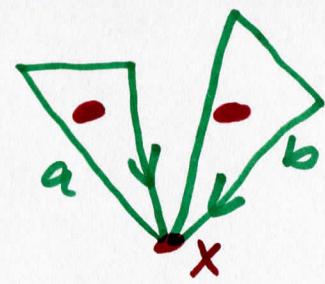
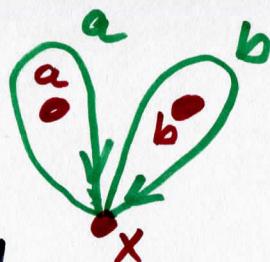


...



$\frac{n}{3}$ copies

$n=2$: Similar, but slightly more complicated

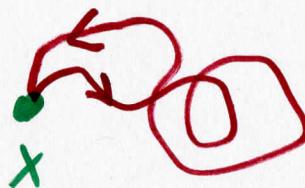


T: $n \geq 2$ fixed:

$$\lim_{m \rightarrow \infty} \frac{cr(n, m)}{m^2} = \infty$$

S: plane - n points \simeq sphere - $n+1$ points

x-loop:



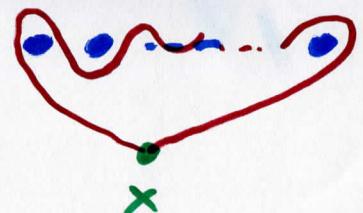
$f(n, k)$: max number of x-loops:

- no two homotopic
- any two cross less than k times
- any one has less than k self-intersections

Juvan-Malnič-Mohar 96: $f(n, k) \leq (n \cdot k)^{c \cdot n \cdot k^2}$ (we: $\leq 2^{2k^{2n}}$)

T: $n \leq 2k$: $f(n, k) \geq 2^{\sqrt{n \cdot k}/3}$

$n \geq 2k$: $f(n, k) \geq \left(\frac{n}{k}\right)^{k-1}$

$k=n$ n holes. For each:  or  $\Rightarrow 2^n$ loops. Any two cross $\leq n-1$ times.

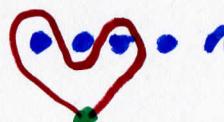
$$\underline{\underline{f(n,n) \geq 2^n}}$$

 $k < n$ only those loops, which have $\leq k-1$ sign changes 

$$2 \cdot \sum_{j=0}^{k-1} \binom{n-1}{j} \geq 2 \cdot \binom{n}{k-1} \geq \left(\frac{n}{k}\right)^{k-1}$$

loops. Any two cross
 $\leq k-1$ times 

$$\underline{\underline{f(n,k) \geq \left(\frac{n}{k}\right)^{k-1}}}$$

 $k > n$ $\sqrt{\frac{k}{n}}$ elementary loops(concatenation) \Rightarrow one loop

$$\sim (2^n)^{\sqrt{\frac{k}{n}}} = 2^{\sqrt{nk}}$$
 loops, any two cross at most

$$\sqrt{k/n} \cdot \sqrt{k/n} \cdot (n-1) < k \text{ times.}$$

$$\underline{\underline{f(n,k) \geq 2^{\sqrt{nk}}}}$$

$$f(n, k) \leq 2^{2k^{2n}} \quad (f(n, k) \leq (nk)^{c \cdot nk^2})$$

1. $f(1, k) \leq 2k+1$

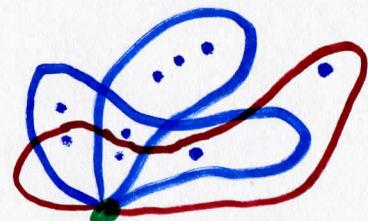
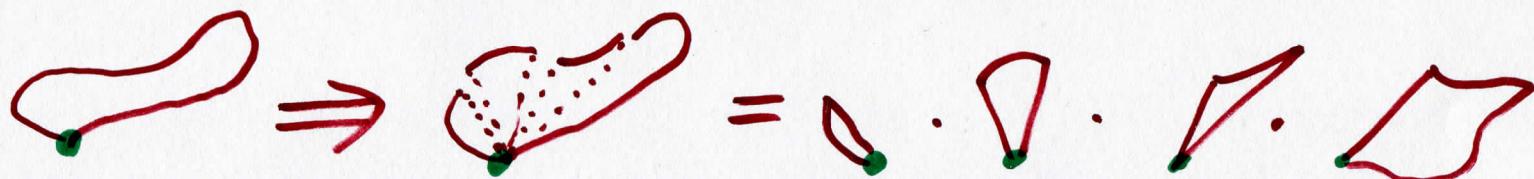


1 hole, winding number
 $\leq k, \geq -k$

2. $n > 1$. Move to sphere: $n+1$ holes.

There are two loops: together each face has
 $< n$ holes.

Other loops \Rightarrow $2k-1$ small loops



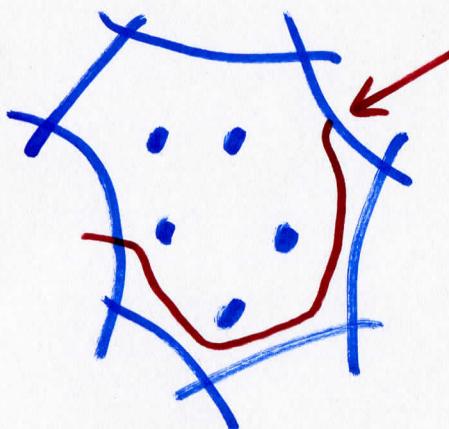
Induction in each component \rightsquigarrow

$$f(n, k) \leq 6k \cdot f(n-1, k)^{2k} \Rightarrow f(n, k) \leq 2^{2k^{2n}}$$

Juvan - Malnič - Mohar '96: Better recursion.

Instead of cutting with two loops:

Take a system of loops that cuts the holes as much as possible. Recursion for each face.



no more cut: very few possibilities

$$f(n, k) \leq (n \cdot k)^{c \cdot n \cdot k^2}$$

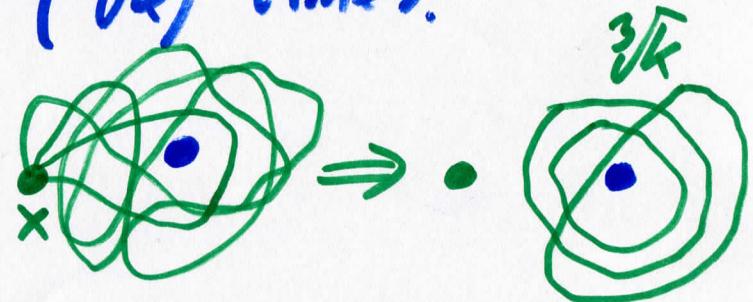
Now prove that $\lim_{m \rightarrow \infty} \frac{cr(n,m)}{m^2} = \infty$

Suppose $m > 2f(n,k) \Rightarrow$ Two loops cross $\geq k$ times (e_i, e'_i) .

Delete them, again, (e_2, e'_2) cross $\geq k$ times

$\Rightarrow \frac{m}{4}$ pairs (e_i, e'_i) : cross $\geq k$ times.

Each pair separate x with a hole many ($\sqrt[3]{k}$) times.



Two pairs that separate the same hole, cross many times!

\Rightarrow many pairwise crossings on the average!

$$cr(n,m) \geq c \cdot m^2 \cdot \frac{\log m}{n^8 \cdot \log \log m}$$