Convex Floating Bodies of Equilibrium

based on joint work with
Han Huang and Boaz Slomka
Dan Florentin, Carsten Schütt and Ning Zhang

Ulam's Problem

Is the Euclidean ball the unique body of uniform density ρ which floats in a liquid in equilibrium in any direction ?

We call such a body Ulam floating body

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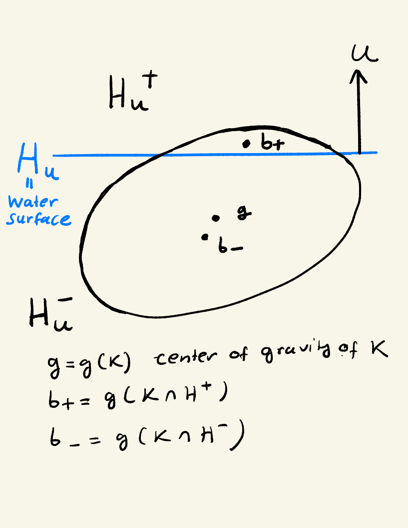
What does *floating in equilibrium in direction u* mean?



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I. SOME BACKGROUND and one RESULT

1. There are **non-symmetric** counterexamples in dimension 2 to Ulam's conjecture by Auerbach for relative density $\rho = \frac{1}{2}$

- $0 \leq \theta \leq 2\pi$, $k \geq 0$, $f(\theta) = -k \cos(3\theta)$
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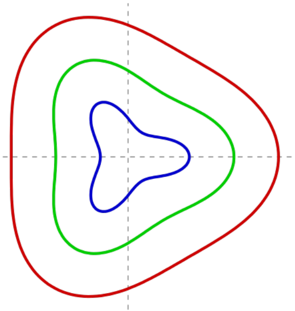

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There are counterexamples in higher dimensions by Wegner. Those are **not** convex - holes are allowed.

Ulam's problem remains mostly open

Theorem 1 (Florentin-Schütt-Werner-Zhang)

Let $K \subset \mathbb{R}^n$ be a symmetric convex body of volume 1 and density $\frac{1}{2}$. If K is an Ulam floating body, then K is a ball.

Remark

In dimension 3 this was proved by Falconer and by Schneider.

II. TOOLS

1. The (convex) floating body K_δ

introduced independently by Barany+Larman and by Schütt+Werner.

Let $\delta \geq 0$ be given.

$$K_\delta = \bigcap_{\left|K \cap H^+_{\delta,u}\right| = \delta |K|} H^-_{\delta,u}$$

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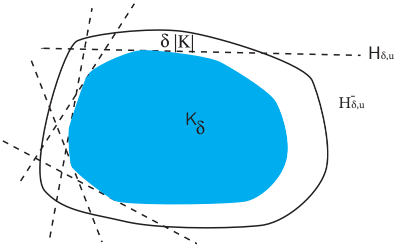

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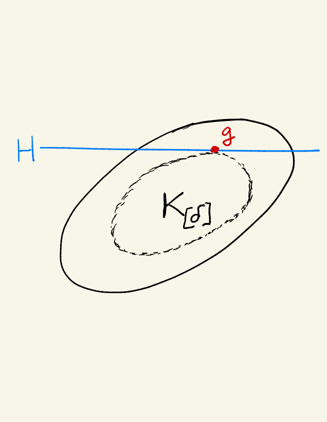
Schütt-Werner
There is δ_0 such such that K_{δ_0} reduces to a point

- If K is symmetric, then $\delta_0 = \frac{1}{2}$
- If K is not symmetric, then $\delta_0 < \frac{1}{2}$ can happen

2. The Dupin floating body $K_{[\delta]}$

$K_{[\delta]}$ is this set contained in K whose boundary is given by the centroids $g = g(K \cap H)$ where the hyperplane H cuts off a set of volume $\delta|K|$ from K



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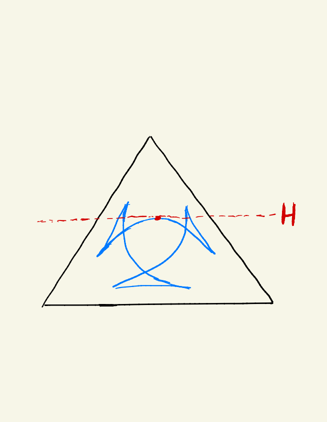

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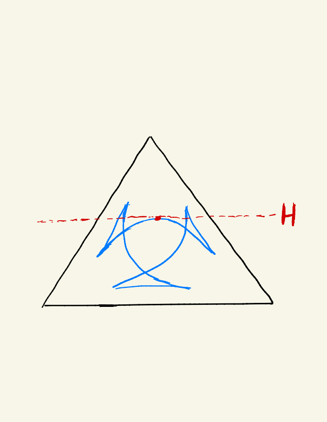

$K_{[\delta]}$ is convex $\implies K_{[\delta]} = K_\delta$

Meyer-Reisner

K symmetric $\implies K_{[\delta]} = K_\delta$

3. Relation between density ρ and the cut off volume $\delta|K|$

WLOG: density of liquid $\rho_L = 1, \quad |K| = 1$

Archimedes law

$$\begin{aligned} \text{weight}(K) &= \text{weight of displaced water} \\ &\Longleftrightarrow \\ \rho|K| = \rho &= \rho_L \cdot (\text{volume of displaced water}) = 1 \cdot (1-\delta) \end{aligned}$$

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$$\iff$$

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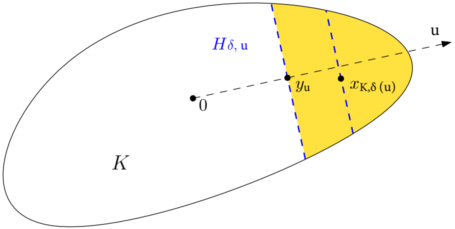
$$\rho = 1 - \delta$$

4. The Metronoid $M_\delta(K)$

introduced by H. Huang and B. Slomka

$M_\delta(K)$ is the body whose boundary consists of the centroids $x_{K,\delta}(u) = g(K \cap H^+_{\delta,u})$ of the floating parts of K



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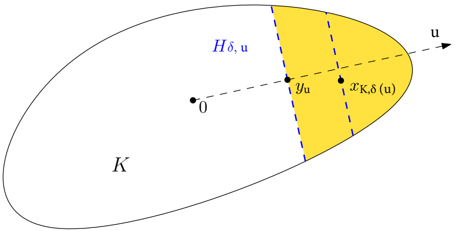

Huang-Slomka

- $M_\delta(K)$ is strictly convex
- $K_{\frac{e-1}{e}\delta} \subset M_\delta(K) \subset K_{\frac{\delta}{e}}$

5. Relation between $M_\delta(K)$ and Ulam floating body

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Proposition (Huang-Slomka-Werner)

$K \subset \mathbb{R}^n$ such that $|K| = 1$ and $g(K) = 0$. Then

$$M_{1-\delta}(K) = -\frac{\delta}{1-\delta} M_\delta(K)$$

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- $\delta = \frac{1}{2}$: $\quad M_{\frac{1}{2}}(K) = -M_{\frac{1}{2}}(K)$, $\quad$ i.e. $M_{\frac{1}{2}}(K)$ is symmetric

Theorem 2 (Florentin-Schütt-Werner-Zhang)

Let $\delta \in \left(0, \frac{1}{2}\right]$ and let $K \subset \mathbb{R}^n$ be a convex body such that K_δ is C^1 or $K_\delta = K_{[\delta]}$ reduces to a point.
K is an *Ulam floating body* if and only if there exists $R > 0$ such that for all $u \in S^{n-1}$ and $v \in S^{n-1} \cap u^\perp$,

$$\int_{K \cap H_{\delta,u}} \langle x, v\rangle^2 - \langle g(K \cap H_{\delta,u}), v\rangle^2 \, dx = \delta |K| \, R.$$

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Remark 1
If K_δ reduces to a point, which wlog we can assume to be 0, then the condition reduces to

$$\int_{K \cap H_{\delta,u}} \langle x, v\rangle^2 \, dx = \delta |K| \, R.$$

III. PROOF OF THEOREM 1

Theorem 1 (Florentin-Schütt-Werner-Zhang)

Let $K \subset \mathbb{R}^n$ be a 0-symmetric convex body of volume 1 and density $\frac{1}{2}$. If K is an Ulam floating body, then K is a ball.

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Theorem 2 $\implies$

K is an Ulam floating body $\iff$

$\forall u \in S^{n-1}, \forall v \in S^{n-1} \cap u^{\perp}: \quad \int_{K \cap H_{\delta,u}} \langle x, v \rangle^2 \, dx = C$

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For $\xi \in S^{n-1}$, let

$$r_K(\xi) = \max\{\lambda \geq 0 : \lambda\xi \in K\}$$

be the radial function of K

Fix $u \in S^{n-1}$. For all $v \in S^{n-1} \cap u^{\perp}$

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Let μ be the normalized Haar measure on $S^{n-2} = S^{n-1} \cap u^{\perp}$.
For all $x \neq 0$

$$\int_{S^{n-2}} \langle x, v \rangle^2 d\mu(v) = 2 \frac{|B_2^{n-2}|}{|S^{n-2}|} \|x\|$$

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0 = \int_{S^{n-1}\cap u^\perp} \left[\frac{2|B_2^{n-2}|}{C(n+1)} r_K(\xi)^{n+1} - 1 \right] d\sigma(\xi)
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Spherical Radon transform $\mathcal{R}$

For a Borel function f on S^{n-1}

$$\mathcal{R}f(u) = \int_{S^{n-1} \cap u^\perp} f(\xi) \, d\sigma(\xi)$$

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Theorem If f is a bounded even Borel function on S^{n-1} such that for all $u \in S^{n-1}$, $\int_{S^{n-1}\cap u^\perp} f(\xi)\, d\sigma(\xi) = 0$, then $f = 0$ σ a.e.

$$0 = \int_{S^{n-1}\cap u^\perp} \left[\frac{2|B_2^{n-2}|}{C(n+1)} \, r_K(\xi)^{n+1} - 1 \right] d\sigma(\xi) = \mathcal{R}(c_n r_K^{n+1} - 1)(u)$$

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r_K is continuous $\implies$ $r_K =$ constant for all $u \implies K$ is a ball