

Quantitative Fractional Helly and (p,q) -Theorems

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joint with

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Part I.
Context and results

Generalizations of Helly's Theorem

Helly's Theorem '23

Let \mathcal{C} be a finite family of convex sets in \mathbb{R}^d . If every $d + 1$ -tuple in \mathcal{C} have **nonempty** intersection, then $\bigcap \mathcal{C}$ is **nonempty**.

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Katchalski, Liu '79: Fractional Helly Theorem

For every $\alpha \in (0, 1]$ there is a $\beta \in (0, 1]$ s.t.

If at least $\alpha \binom{n}{d+1}$ of the $d + 1$ -tuples from \mathcal{C} have **nonempty** intersection, then $\exists \mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| \geq \beta |\mathcal{C}|$ s.t. $\cap \mathcal{C}'$ is **nonempty**.

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Alon, Kleitman '92: (p, q) -Theorem

For every $p \geq q \geq d + 1$ there is a $H_d(p, q)$ s.t.
If among any p members of \mathcal{C} there are q whose intersection is **nonempty**,
then there is a set of at most H points **intersecting** all the members of \mathcal{C} .

Quantitative Helly Theorem

Bárány, Katchalski and Pach '82: Quantitative Volume Theorem

Let \mathcal{C} be a finite family of convex bodies in \mathbb{R}^d such that any $2d$ of them have intersection of volume at least 1.

Then $\cap \mathcal{C}$ is of volume at least d^{-cd^2} .

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John's Theorem '48

If $K \subset \mathbb{R}^d$ is a convex body, then it contains a unique ellipsoid, $John(K)$ of maximal volume. Moreover, $K \subset d \cdot John(K)$.

Quantitative Fractional Helly Theorems

J., Naszódi: QFH with large intersections

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- ▶ It would be nice to have $2d$ instead of $3d+1$.
- ▶ d^{-cd} in place of d^{-cd^2} ?

Quantitative (p,q) Theorem

Quantitative transversal number

A set T of ellipsoids of **volume v** is a **quantitative v -transversal** of \mathcal{C} , if every $C \in \mathcal{C}$ contains at least one ellipsoid from T .

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For every $p \geq q \geq 3d + 1$, there is $H_d(p, q)$ s.t.:

If \mathcal{C} is a finite family of convex bodies in \mathbb{R}^d ,

every member of \mathcal{C} contains an ellipsoid of **volume 1**, and

among any p of them there are q whose intersection also contains an ellipsoid of **volume 1**.

Then, \mathcal{C} has a quantitative d^{-cd^2} -transversal of cardinality at most H .

Part II.
Some word on proofs

Structure of the proof of the Quantitative (p,q) -Theorem

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- ▶ Alon, Kalai, Meshulam and Matousek '02: a Fractional Helly Theorem implies a (p, q) -Theorem for hypergraphs.

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- ▶ Alon, Kalai, Meshulam and Matousek '02: a Fractional Helly Theorem implies a (p, q) -Theorem for hypergraphs.
- ▶ But we have **two different hypergraphs** in the assumption and the conclusion.

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First step

QFH with
FH number $\frac{d(d+3)}{2}$

Q Tverberg
[Sarkar, Xue and
Soberón '19]

\implies

Q Selection
Lemma

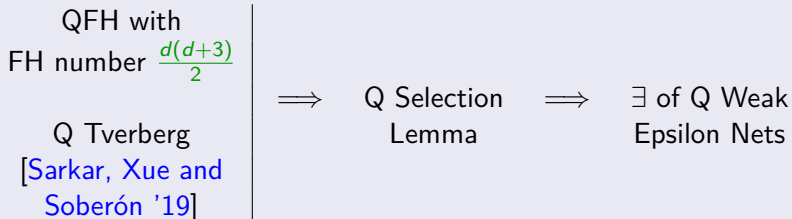
\implies

\exists of Q Weak
Epsilon Nets

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Second step



Idea of the proof of QFH with large intersections

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- ▶ **Damásdi, Földvári and Naszódi '19**: A similar proof of a Quantitative Colorful Helly Theorem with $\frac{d(d+3)}{2}$ colorclasses.

Proof of QFH with small intersections - Part 1

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- ▶ Let $\mathcal{C} = \{C_1, \dots, C_n\}$. We have at least $\alpha \binom{n}{3d+1}$ index sets $I \subset [n]$ with size $3d+1$ for whom $\cap_{i \in I} C_i$ contains an ellipsoid of volume 1. We call these index sets **good**.

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- ▶ **Quantitative Helly**: for every **good** index set I there is a subset $S \subset I$ with size $2d$ for whom $\text{vol}(\text{John}(\cap_{i \in S} C_i)) \leq d^{c'd} \text{vol}(\text{John}(\cap_{i \in I} C_i))$.

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- ▶ call S the **seed** of I .

Proof of QFH with small intersections - Part 2

- ▶ There is an S who is the **seed** of at least

$$\frac{\alpha\binom{n}{3d+1}}{\binom{n}{2d}} \geq \gamma \binom{n}{d+1}$$

of the **good** index sets. Call them $I_1, \dots, I_{\gamma\binom{n}{d+1}}$.

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- ▶ **Lemma₁** : since $\mathcal{E}_j \subseteq \bigcap_{i \in S} C_i$, there is a $v_j \in \mathbb{R}^d$ such that $d^{-cd}\mathcal{E} + v_j \subseteq \mathcal{E}_j$.

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