Quantitative Fractional Helly and (p,q)-Theorems

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joint with

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Part I.
Context and results

Generalizations of Helly's Theorem

Helly's Theorem '23

Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$. If every $d + 1$-tuple in $\mathcal{C}$ have nonempty intersection, then $\cap\mathcal{C}$ is nonempty.

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Katchalski, Liu '79: Fractional Helly Theorem

For every $\alpha \in (0,1]$ there is a $\beta \in (0,1]$ s.t.
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Alon, Kleitman '92: (p,q)-Theorem

For every $p \geq q \geq d+1$ there is a $H_d(p,q)$ s.t.
If among any p members of $\mathcal{C}$ there are q whose intersection is nonempty,
then there is a set of at most H points intersecting all the members of $\mathcal{C}$.

Quantitative Helly Theorem

Bárány, Katchalski and Pach '82: Quantitative Volume Theorem

Let $\mathcal{C}$ be a finite family of convex bodies in $\mathbb{R}^d$ such that any 2d of them have intersection of volume at least 1.
Then $\cap\mathcal{C}$ is of volume at least d^{-cd^2}.

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John's Theorem '48

If $K \subset \mathbb{R}^d$ is a convex body, then it contains a unique ellipsoid, $John(K)$ of maximal volume. Moreover, $K \subset d \cdot John(K)$.

Quantitative Fractional Helly Theorems

J., Naszódi: QFH with large intersections

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- It would be nice to have $2d$ instead of $3d+1$.
- d^{-cd} in place of d^{-cd^2}?

Quantitative (p,q) Theorem

Quantitative transversal number

A set T of ellipsoids of volume v is a **quantitative v-transversal** of $\mathcal{C}$, if every $C \in \mathcal{C}$ contains at least one ellipsoid from T.

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For every $p \geq q \geq 3d+1$, there is $H_d(p,q)$ s.t.:
If $\mathcal{C}$ is a finite family of convex bodies in $\mathbb{R}^d$,
every member of $\mathcal{C}$ contains an ellipsoid of volume 1, and
among any p of them there are q whose intersection also contains an ellipsoid of volume 1.
Then, $\mathcal{C}$ has a quantitative d^{-cd^2}-transversal of cardinality at most H.

Part II.
Some word on proofs

Structure of the proof of the Quantitative (p,q)-Theorem

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- But we have two different hypergraphs in the assumption and the conclusion.

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First step

QFH with FH number $\frac{d(d+3)}{2}$ Q Tverberg [Sarkar, Xue and Soberón '19]	$\Longrightarrow$	Q Selection Lemma	$\Longrightarrow$	$\exists$ of Q Weak Epsilon Nets

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Second step

QFH with FH number $3d + 1$	$\Longrightarrow$	Bounded Q Fractional d^{-cd^2}-Transversal Number

Idea of the proof of QFH with large intersections

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- Damásdi, Földvári and Naszódi '19: A similar proof of a Quantitative Colorful Helly Theorem with $\frac{d(d+3)}{2}$ colorclasses.

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- Let $\mathcal{C} = \{C_1, \ldots, C_n\}$. We have at least $\alpha\binom{n}{3d+1}$ index sets $I \subset [n]$ with size $3d+1$ for whom $\cap_{i\in I} C_i$ contains an ellipsoid of volume 1. We call these index sets good.

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- **Quantitative Helly**: for every good index set I there is a subset $S \subset I$ with size $2d$ for whom $vol(John(\cap_{i \in S} C_i)) \leq d^{c'd} vol(John(\cap_{i \in I} C_i))$.

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- call S the seed of I.

Proof of QFH with small intersections - Part 2

- There is an S who is the seed of at least

$$\frac{\alpha\binom{n}{3d+1}}{\binom{n}{2d}} \geq \gamma\binom{n}{d+1}$$

of the good index sets. Call them $I_1, \ldots, I_{\gamma\binom{n}{d+1}}$.

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