DOMES OVER CURVES

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PLAN

1. Kenyon's problem
2. Existence of domes
3. Non-existence of domes
4. Open problems

A closed piecewise linear curve is called integral if it is comprised of segments of integer length.

Let $S \subset \mathbb{R}^3$ be a PL-surface realized in $\mathbb{R}^3$ with the boundary $\partial S = \gamma$, and with all facets comprised of unit equilateral triangles. In this case we say that S is a dome over γ and that γ can be domed.

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Can every integral closed curve $\gamma \subset \mathbb{R}^3$ be domed?

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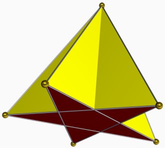

REAL-LIFE DOMES

Buckminster Fuller's Dome over Manhattan (1960)

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Geodesic domes: Montreal Biosphère, Buckminster Fuller's dome, Geometrica's Freedome

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Lemma

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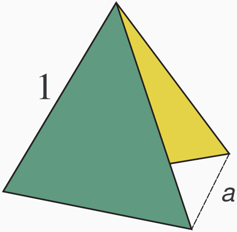

Theorem (G.-Pak, 2020)

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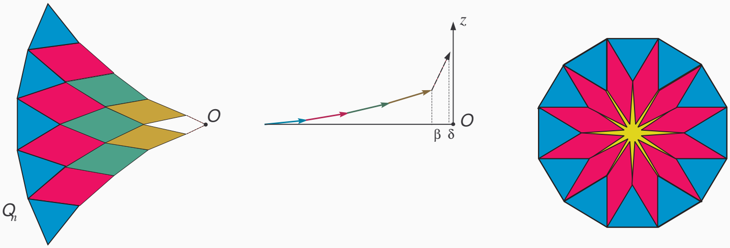
Add triangles forming a small angle θ with the plane.

Add almost planar rhombi until they reach the center.
Continuously vary θ to make rhombi meet above the center.



Theorem (G.-Pak, 2020)

For every integral curve $\gamma \subset \mathbb{R}^3$ and $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$ of equal length, such that $|\gamma, \gamma'|_F < \varepsilon$ and γ' can be domed.

Here $|\gamma, \gamma'|_F$ is the Fréchet distance $|\gamma, \gamma'|_F = \max_{1 \le i \le n} |v_i, v_i'|$.

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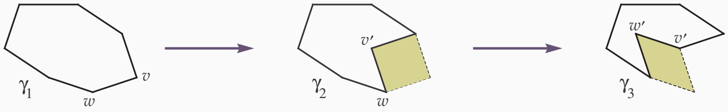
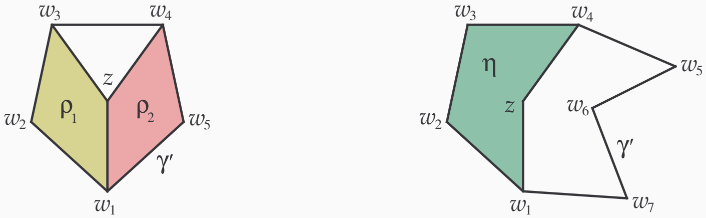
Step 1. Curve $\to$ Almost planar curve via flips

Step 2. Almost planar curve $\to$ Almost planar curve with distances $\le \sqrt{5/4}$ via flips (use Steinitz+Bergström)



Step 3. Split the curve into a pentagon and a shorter curve. Use an ad-hoc construction for pentagons and the inductive hypothesis for a shorter curve.



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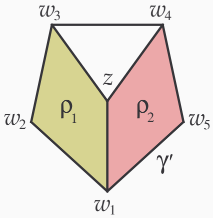

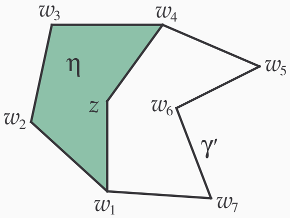

Step 4. Fix the combinatorial decomposition. Slightly alter the curve to ensure that everything is generic and apply the Rhombus lemma at all steps.

Theorem (G.-Pak, 2020)

Let $\rho(a, b) \subset \mathbb{R}^3$ be a rhombus with diagonals $a, b > 0$. Suppose $\rho(a, b)$ can be domed. Then there is a nonzero polynomial $P \in \mathbb{Q}[x, y]$, such that $P(a^2, b^2) = 0$.

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Let $s \notin \overline{\mathbb{Q}}$, and let s^2 and t^2 be algebraically dependent with the minimal polynomial $Q(s^2, t^2) = 0$. Suppose $Q \in \overline{\mathbb{Q}}[x,y]$ is given by

$$Q(x,y) \, = \, x^k y^{m-k} + \sum_{i+j<m} c_{ij} x^i y^j \, ,$$

for some $0 \le k \le m$. Then $\rho(s,t)$ cannot be domed.

K is a simplicial connected pure 2-dimensional complex with a free action of the group $G = \mathbb{Z} \oplus \mathbb{Z}$ with generators a and b.

θ : K $\to \mathbb{R}^3$ is linear on each simplex of K and equivariant with respect to the action of $\mathbb{Z} \oplus \mathbb{Z}$, such that a and b act by translations with vectors α and β.

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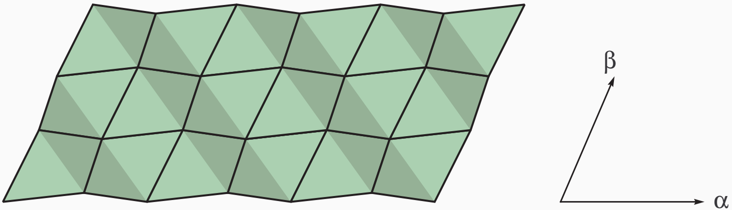

Let $\mathcal{G}(\mathrm{K})$ be the set of all possible Gram matrices formed by vectors α and β for all doubly periodic triangular surfaces (K, θ). Denote $g_{11} = |\alpha|^2$, $g_{12} = g_{21} = \alpha \cdot \beta$, $g_{22} = |\beta|^2$.

Theorem (A. Gaifullin – S. Gaifullin, 2014)

For K homeomorphic to $\mathbb{R}^2$, there is a one-dimensional real affine algebraic subvariety containing $\mathcal{G}(\mathrm{K})$. In particular, the entries of each Gram matrix G from $\mathcal{G}(\mathrm{K})$ satisfy a system of two non-trivial polynomial equations with integer coefficients:

$$\begin{cases} P(g_{11}, g_{12}, g_{22}) = 0 \\ Q(g_{11}, g_{12}, g_{22}) = 0. \end{cases}$$

Proposition

Suppose a rhombus $\gamma = \rho(s, t)$ can be domed by a surface homeomorphic to a disk. Then there exists a polynomial $F \in \mathbb{Q}[x, y]$, such that $F(s^2, t^2) = 0$.

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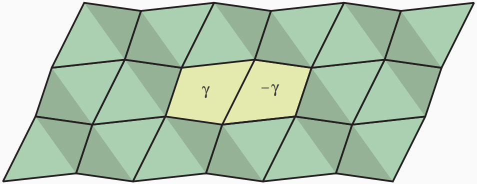
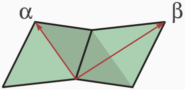
Proof.

Attach a dome to each copy of γ and $-\gamma$. The resulting surface is doubly periodic with vectors α and β formed by the diagonals of γ. By the G-G theorem, either P or Q is not trivial when $\alpha \cdot \beta = 0$.





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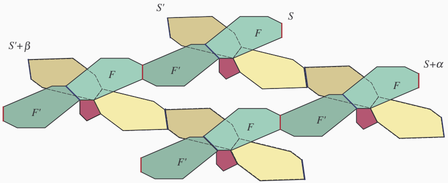

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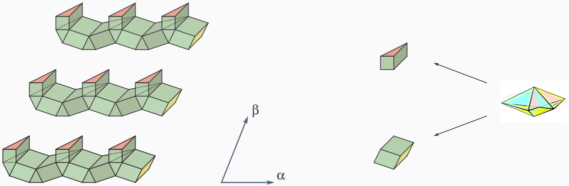

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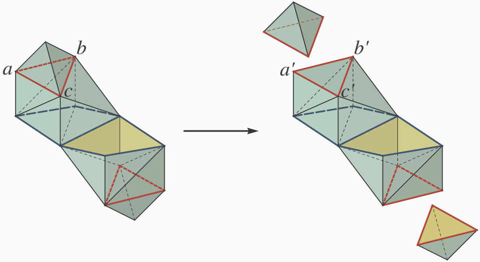
Conclusion: the general version of the G-G theorem does not work. We need a different result.

- Places of fields.
- Induction over the combinatorial structure of a surface via surface surgeries and flips. (Here it is important that the surface is based on rhombi).
- Lemma by Connelly-Sabitov-Walz (1997) on the existence of a diagonal where a place is finite.
- Construction of a polynomial similar to Gaifullin-Gaifullin.



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A triangle Δ with side lengths $(2, 2, 1)$ cannot be domed.

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Proof.

Four copies of triangles $(3, 7, 7)$ and $(4, 7, 7)$ form a Bricard octahedron. □

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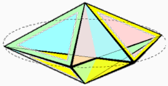

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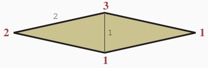
Let $\rho_{\Diamond} = [\mathrm{uvwx}] \subset \mathbb{R}^3$ be the planar rhombus with all side lengths 2 and $|\mathrm{xv}| = 1$. Then there is a coloring $\chi : \mathbb{R}^3 \to \{1, 2, 3\}$ with no rainbow unit triangles, and such that $\chi(\mathrm{u}) = \chi(\mathrm{v}) = 1$, $\chi(\mathrm{w}) = 2$, $\chi(\mathrm{x}) = 3$.



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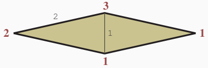

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Conjecture

Let $\gamma = [v_1 \ldots v_n]$ be an integral curve which can be domed, where $n \geq 5$. Denote by $d_{ij} = |v_i v_j|$ the diagonals of γ. Then there is a nonzero polynomial $P \notin \mathrm{CM}_n$, such that $P\big(d_{1,3}^2, d_{1,4}^2, \ldots, d_{n-2,n}^2\big) = 0$, where CM_n is the ideal generated by all Cayley-Menger determinants.

THANK YOU!