Helly numbers for crystals and cut-and-project sets

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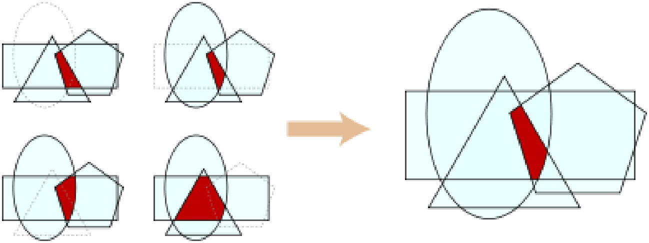
HELLY THEOREM

Theorem (Helly, 1923)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If any $d+1$ sets from $\mathcal{F}$ have a non-empty intersection, then all sets from $\mathcal{F}$ have a non-empty intersection.



Helly-type theorem for lattices

Theorem (Doignon, 1973)

Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$. If any 2^d sets from $\mathcal{F}$ intersect at an integer point, then all sets from $\mathcal{F}$ intersect at an integer point.

In both theorems the numbers $d+1$ and 2^d can't be decreased.

Helly number for a set

Definition

Fix a set S in $\mathbb{R}^d$.

Let n be the smallest integer number such that the following condition hold for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$. If any n sets from $\mathcal{F}$ intersect at a point of S, then all sets from $\mathcal{F}$ intersect at a point of S.

This number n is called the **Helly number of S**, or just **S-Helly number**, or $h(S)$.

If a number n from the definition doesn't exist, then $h(S) = \infty$.

- $h(\mathbb{R}^d) = d + 1$.
- $h(\mathbb{Z}^d) = h(d\text{-dimensional lattice}) = 2^d$.
- $h(\mathbb{S}^{d-1}) = \infty$.

Discrete and Delone sets

Definition

Set S in $\mathbb{R}^d$ is called a **discrete set** if every ball in $\mathbb{R}^d$ contains only finitely many points of S.

Definition

Set S in $\mathbb{R}^d$ is called a **Delone set** if

- distance between two different points of S is greater than some fixed positive number;
- radius of a ball in $\mathbb{R}^d$ without points of S is less than some fixed positive number.

Sometimes Delone sets are called separated nets which are **uniformly discrete** and **relatively dense**.

Helly numbers for discrete sets

Let S be a discrete set in $\mathbb{R}^d$ and $n(S)$ be the largest number of vertices of an **empty S-polytope**.

Definition

The **convex** polytope P is an empty S-polytope if

- all vertices of P are from S, and
- P does not contain other points from S.

Lemma (Hoffman, 1979)

$h(S) = n(S)$ or $h(S) = \infty$ if $n(S)$ does not exist.

Helly numbers for Delone sets. Finite and infinite

Example

For $\mathbb{Z}^d$, the unit cube $[0,1]^d$ has 2^d vertices and is an empty $\mathbb{Z}^d$-polytope.

Any set of at least $2^d + 1$ points from $\mathbb{Z}^d$ contains two points from one parity class and does not give an empty $\mathbb{Z}^d$-polytope.

Example

For every $n \geq 3$ we can find a convex n-gon with vertices in the lattice $\mathbb{Z}^2$. Using an appropriate $GL_2(\mathbb{Z})$ transformation we can transform this polygon into a "very thin" (and long) lattice polygon P_n.

Placing copies of P_n "very far apart" and removing lattice points inside each P_n we get a Delone set S with $h(S) = \infty$.

Discrete sets with finite Helly number

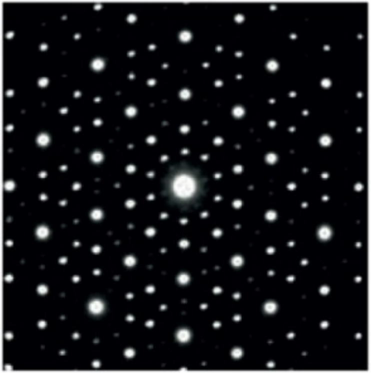
What are discrete or Delone sets with finite Helly number?

Theorem (De Loera, La Haye, Oliveros, Roldán-Pensado)

- *If* $S = \mathbb{Z}^2 \setminus L$ *where* L *is a sublattice of* $\mathbb{Z}^2$*, then* $h(S) \leq 6$*;*
- *If* $S = \mathbb{Z}^d \setminus (L_1 \cup \ldots \cup L_k)$ *where each* L_i *is a (shifted) sublattice of* $\mathbb{Z}^d$*, then* $h(S) \leq C_k 2^d$*.*

Crystals and cut-and-project sets

- repetitive clusters;
- periodic or quasi-periodic;
- "good" x-ray diffraction picture.



Crystals and crystallographic groups

Definition

Crystallographic group is a discrete subgroup of isometries with bounded fundamental domain.

Definition

Crystal is a union of finitely many orbits of a crystallographic group.

Theorem (Bieberbach, 1911-12)

Every crystallographic group in $\mathbb{R}^d$ contains a subgroup of finite index isomorphic to $\mathbb{Z}^d$.

Definition

k-crystal in $\mathbb{R}^d$ is a union of k translations of the same lattice.

Helly number for crystals

Theorem

If S is a k-crystal in $\mathbb{R}^d$, then $h(S) \leq k2^d$.

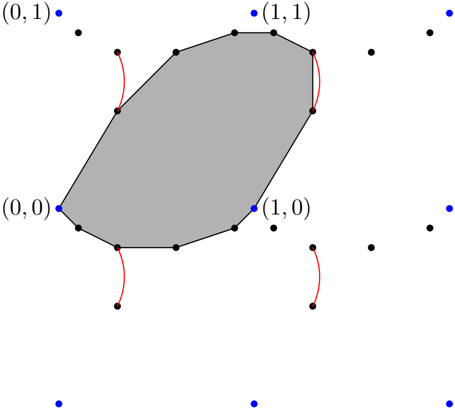
Proof.

Let P be an empty S-polytope with at least $k2^d + 1$ vertices. Then one copy of the corresponding lattice contains at least $2^d + 1$ vertices of P and therefore is not empty. □

Two-dimensional crystals

Theorem

If S is a k-crystal in $\mathbb{R}^2$, then $h(S) \le k+6$; tight for $k \ge 6$.



Two-dimensional crystals

Theorem

If S is a k-crystal in $\mathbb{R}^2$, then $h(S) \leq k + 6$; tight for $k \geq 6$.

Proof.

Let P be an empty S-polygon with maximal number of vertices. If N is the maximal number of vertices of P from one copy of the lattice, then we can bound the number of lattice copies with more than one point in P depending on N. □

General bounds of $h(S)$ for k-crystals.

k	1	2	3	4	5	≥ 6
max. $h(S)$	4	6	7	9	10	$k+6$

d-dimensional crystals

Theorem

For $d \geq 2$ and for $k \geq 6$ there exists a d-dimensional k-crystal S with

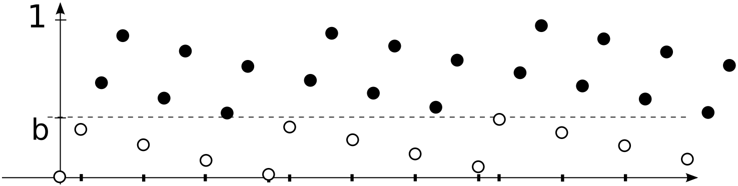
$$h(S) \geq (k+6)2^{d-2}$$

If $h_{d,k}$ is the maximal Helly number among all d-dimensional k-crystals, then

$$(k+6)2^{d-2} \leq h_{d,k} \leq k2^d = 4k \cdot 2^{d-2}$$

provided $d \geq 2$ and $k \geq 6$.

Cut-and-project sets, pictorial definition



Cut-and-project sets, formal definition

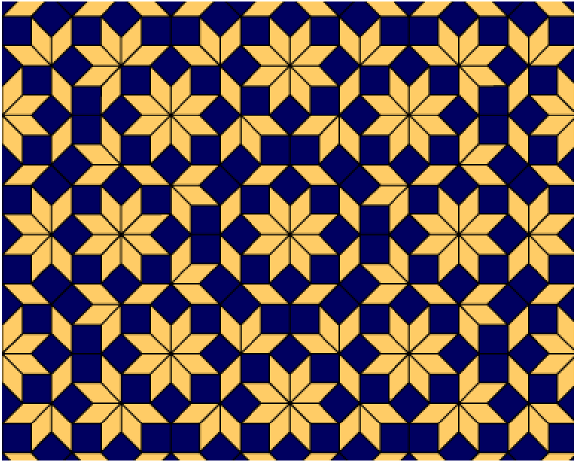
Definition

- Λ is a $(d+k)$-dimensional lattice in $\mathbb{R}^d \times \mathbb{R}^k$;
- W is a compact set in $\mathbb{R}^k$ such that closure of the interior of W is W, **the window**;
- π_1 and π_2 are projections on $\mathbb{R}^d$ and $\mathbb{R}^k$. The projection $\pi_1|_\Lambda$ is injective and the projection $\pi_2(\Lambda)$ is dense.

Then $V = V(\mathbb{R}^d, \mathbb{R}^k, \Lambda, W) = \{\pi_1(\mathbf{x}) \mid \mathbf{x} \in \Lambda,\ \pi_2(\mathbf{x}) \in W\}$ is **cut-and-project set**.

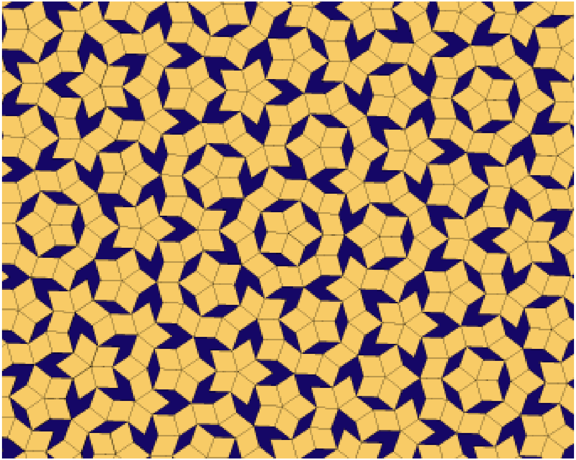
$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{R}^k & \xrightarrow{\pi_2} & \mathbb{R}^k \\ \cup & & \cup & & \cup \\ V & & \Lambda & & W \end{array}$$

Cut-and-project sets: examples



Ammann-Beenker tiling can be constructed using a two-dimensional window and four-dimensional lattice

Cut-and-project sets: examples



Penrose tiling can be constructed using a three-dimensional window and five-dimensional lattice

Helly numbers for cut-and-project sets

Theorem

If $V = V(\mathbb{R}^d, \mathbb{R}^k, \Lambda, W)$ is a cut-and-project set with a **convex** *window W, then $h(V) \leq 2^{d+k}$.*

Proof.

- Suppose P is an empty V-polytope with at least $2^{d+k}+1$ vertices;
- π_1-preimages of the vertices of P are points of the lattice Λ;
- Due to Doignon's theorem, there is an additional lattice point $\mathbf{x}$ in the convex hull of preimages;
- Due to convexity of W, point $\pi_1(\mathbf{x})$ is in V and in P.

□

Further questions: Crystals

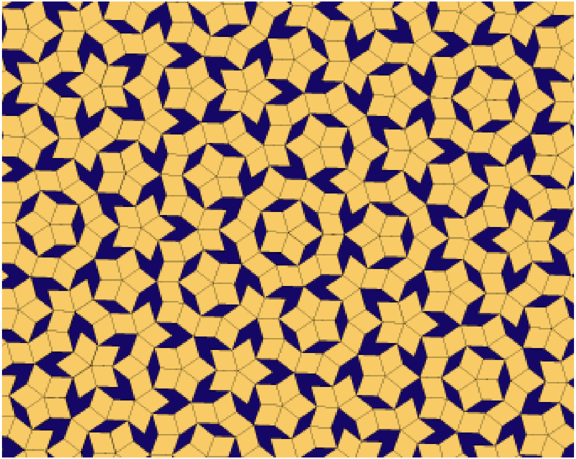
Question

What is the exact value for $h_{d,k}$?

$$(k+6)2^{d-2} \leq h_{d,k} \leq k2^d$$

Further questions: Cut-and-project sets

The upper bound $h(V) \leq 2^{d+k}$ for d-dimensional cut-and-project sets with k-dimensional window looks very non-optimal.



Further questions: Cut-and-project sets

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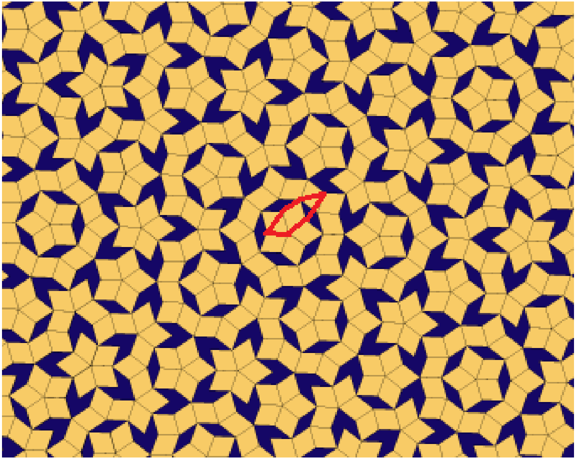


Further questions: Cut-and-project sets

The upper bound $h(V) \leq 2^{d+k}$ for d-dimensional cut-and-project sets with k-dimensional window looks very non-optimal.

Conjecture

$$h(\textit{vertices of a Penrose tiling}) = 6$$

Conjecture (weaker version)

$$h(\textit{vertices of a Penrose tiling}) \leq 10$$

Further questions

Question

What about Helly numbers for discrete sets with weaker or different structure?

- *Cut-and-project sets with non-convex windows;*

Further questions

Question

What about Helly numbers for discrete sets with weaker or different structure?

- *Cut-and-project sets with non-convex windows;*
- *FLC sets.*

Definition

For $\mathbf{x} \in X$, the set of points of X at distance at most r from $\mathbf{x}$ is called the ***r*-cluster** of $\mathbf{x}$.

Definition

X is called a set with **finite local complexity** if for every r it has only finitely many **non-equivalent** r-clusters.
$N(r)$, the number of equivalence classes of r-clusters, is called the **cluster counting function**.

Further questions: Meyer sets

Definition

A Delone set X is called a **Meyer set** if the set of differences $X - X$ is a Delone set as well.

Theorem (Lagarias, 1999)

A Delone set is a Meyer set if and only if it is a subset of cut-and-project set.

Theorem

Every cut-and-project set contains a Delone set X with $h(X) = \infty$.

Further questions: Repetitive sets

Definition

The Delone set X is called **repetitive** if there exists a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that every ball of radius $f(r) + r$ contains every r-cluster of X.

Definition

The Delone set X is called **linearly repetitive** if f can be chosen to be linear, so there exists C such that every ball of radius Cr contains every r-cluster of X.

Theorem (Lagarias, Pleasants, 2003)

Linearly repetitive Delone sets have well-defined frequencies for every local cluster.

Further questions: Repetitive sets

Question

Does every repetitive Delone set have a finite Helly number?

Question

Does every linearly repetitive Delone set have a finite Helly number?

Further questions: Repetitive sets

Question

Does every repetitive Delone set have a finite Helly number?

Question

What about **densely repetitive** *Delone sets? For these sets* $f(r) = O(N(r))^{1/d}$.

Question

Does every linearly repetitive Delone set have a finite Helly number?

Further questions: Hyperbolic plane

Question

Does every crystallographic set in $\mathbb{H}^2$ have a finite Helly number?

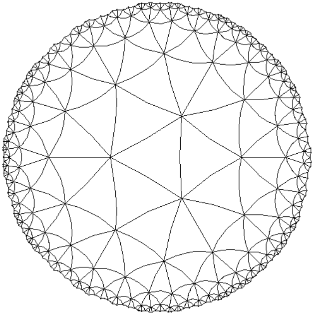
Further questions: Hyperbolic plane

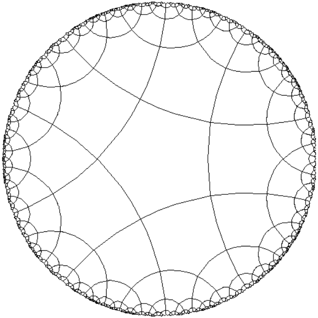
Question

Does every crystallographic set in $\mathbb{H}^2$ have a finite Helly number?

Question

*Which **regular** sets in $\mathbb{H}^2$ have finite Helly number?*





Further questions: Hyperbolic plane

Question

Is there a Delone set in $\mathbb{H}^2$ with finite Helly number?

Thank you!