# A remark on the minimal dispersion

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**Equivalently:** Given integers $n, d \geq 1$ what is the largest $\varepsilon > 0$ such that for any $n$ points in the unit $d$-dimensional cube $[0,1]^d$ there exists an axis-parallel box of volume $\varepsilon$ containing none of these points?

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Such an $\varepsilon$ is called dispersion (of the cube) or *minimal dispersion* and denoted by

$$\mathrm{disp}^*(n, d).$$

# Notations.

Consider the set of all axis parallel boxes contained in the cube $[0, 1]^d$,

$$\mathcal{R}_d := \left\{ \prod_{i=1}^{d} I_i \;\middle|\; I_i = [a_i, b_i) \subset [0, 1] \right\}.$$

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In this talk it will be more convenient to work with its inverse, the function

$$N(\varepsilon, d) = \min\{n \in \mathbb{N} \mid \mathrm{disp}^*(n, d) \leq \varepsilon\}.$$

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The first bound showing that the minimal dispersion grows with the dimension was obtained by Aistleitner–Hinrichs–Rudolf (**17**):

$$N(\varepsilon, d) \ge (1 - 4\varepsilon)\, \frac{\log_2 d}{4\varepsilon}.$$

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$$(1-4\varepsilon)\,\frac{\log_2 d}{4\varepsilon} \le N(\varepsilon,d) \le \frac{Cd^2\log d}{\varepsilon}. \qquad (1)$$

Thus, when the dimension $d$ is fixed and $\varepsilon \to 0$ the problem is essentially solved:

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Estimate (2) is better than the upper bound in (1) in the regime $\varepsilon \ge d^{-Cd}$.

# Known results.

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Their upper bound is better in the regime

$$\varepsilon \geq \frac{C\,(\log_2 d)^2}{d}.$$

# Known results.

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Does not grow when $d \to \infty$. Recall, for $\varepsilon < 1/4$, we have $N(\varepsilon, d) \geq C_\varepsilon \log_2 d$.

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**Problem.** What is $N(1/4, d)$? How does $N(\varepsilon, d)$ behave when $\varepsilon \to (1/4)^{\pm}$.

# Upper bounds: summary

Till very recently:

$$N(\varepsilon, d) \leq \begin{cases} \dfrac{C\ \ln d}{\varepsilon^2} \ln^2\left(\dfrac{1}{\varepsilon}\right), & \text{if} \quad \varepsilon \geq \dfrac{\ln^2 d}{d}, \\ \dfrac{C\, d}{\varepsilon} \ln\left(\dfrac{1}{\varepsilon}\right), & \text{if} \quad \dfrac{\ln^2 d}{d} \geq \varepsilon \geq \exp(-C^d), \\ \dfrac{C^d}{\varepsilon}, & \text{if} \quad \varepsilon \leq \exp(-C^d). \end{cases}$$

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Using the Bukh–Chao result:

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# New bounds.

## Theorem (small $\varepsilon$)

*Let* $d \geq 2$ *and* $\varepsilon \leq 1/2$. *Then*

(i) $$N(\varepsilon, d) \leq \frac{C \ln d}{\varepsilon} \ln\left(\frac{1}{\varepsilon}\right),$$ *provided that* $\varepsilon \leq \exp(-d)$,

(ii) $$N(\varepsilon, d) \leq \frac{C\, d}{\varepsilon} \ln\ln\left(\frac{2}{\varepsilon}\right),$$ *provided that* $\varepsilon \geq \exp(-d)$.

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*Moreover, the random choice of points uniformly distributed on* $[0,1]^d$ *works.*

Thus, for $\varepsilon \leq \exp(-d)$ we have

$$\frac{\ln d}{6\varepsilon} \leq N(\varepsilon, d) \leq \frac{C \ln d}{\varepsilon} \ln\left(\frac{1}{\varepsilon}\right),$$

In $(ii)$ the improvement is only in substitution of $\ln(1/\varepsilon)$ with $\ln\ln(1/\varepsilon)$.

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Hinrichs–Krieg–Kunsch–Rudolf (**20**): for a random choice of points

$$\max\left\{\frac{c}{\varepsilon}\ln\left(\frac{1}{\varepsilon}\right),\ \frac{d}{2\varepsilon}\right\} \le N_{ran}(\varepsilon, d)$$

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## Theorem (relatively large $\varepsilon$)

*Let* $d \ge 2$ *and* $\frac{\ln d}{d} \le \varepsilon \le 1/2$. *Then*

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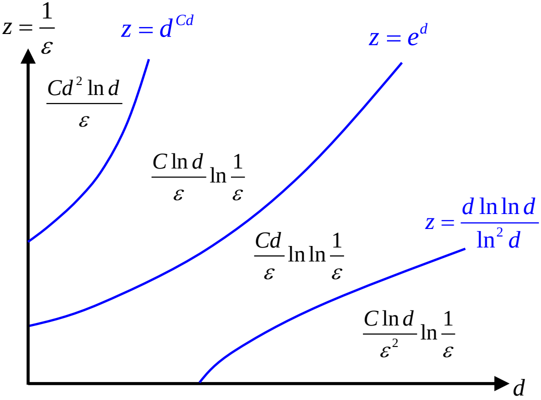
The prove also uses random points, but one needs to adjust the distribution.

# State of the art.

$$N(\varepsilon, d) \leq \begin{cases} \frac{C\,\ln d}{\varepsilon^2}\,\ln\left(\frac{1}{\varepsilon}\right), & \text{if } \varepsilon \geq \frac{\ln^2 d}{d \ln\ln(2d)}, \\ \frac{C\,d}{\varepsilon}\,\ln\ln\left(\frac{1}{\varepsilon}\right), & \text{if } \frac{\ln^2 d}{d\ln\ln(2d)} \geq \varepsilon \geq e^{-d}, \\ \frac{C\ln d}{\varepsilon}\,\ln\left(\frac{1}{\varepsilon}\right), & \text{if } e^{-d} \geq \varepsilon \geq \exp(-Cd\ln d), \\ \frac{C\,d^2\ln d}{\varepsilon}, & \text{if } \varepsilon \leq \exp(-Cd\ln d). \end{cases}$$



# Some ideas of the proof.

Consider a set $P$ of random points independently and uniformly distributed in the cube. To show that (with good probability) every box of volume $\varepsilon$ contains at least one point from $P$, we need to construct a set $\mathcal{N}$ of "test boxes." It should satisfy

$$\textit{each rectangle in } \mathcal{N} \textit{ contains a point from } P \quad \Longrightarrow$$

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as the sum of "bad" probabilities of individual events,

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Rudolf used the concept of $\delta$-cover to construct $\mathcal{N}$ (and bounds due to Gnewuch).

# Some ideas of the proof.

Consider a set $P$ of random points independently and uniformly distributed in the cube. To show that (with good probability) every box of volume $\varepsilon$ contains at least one point from $P$, we need to construct a set $\mathcal{N}$ of "test boxes." It should satisfy

$$\textit{each rectangle in } \mathcal{N} \textit{ contains a point from } P \quad \Longrightarrow$$

$$\textit{each rectangle in } \mathcal{R}_d \textit{ of volume at least } \varepsilon \textit{ contains a point from } P.$$

Standardly, using union bound, one estimates the probability of the "bad" event

*there exists a rectangle in $\mathcal{N}$ containing no points from $P$*

as the sum of "bad" probabilities of individual events,

*a given rectangle in $\mathcal{N}$ containing no points from $P$.*

Individual bounds are simple – volume computations.

Thus the main difficulty is to construct the set $\mathcal{N}$ of not too large cardinality.

Rudolf used the concept of $\delta$-cover to construct $\mathcal{N}$ (and bounds due to Gnewuch).

We use a different approach, which fits better this problem.

# Some ideas of the proof.

Denote

$$\mathcal{B}_{\varepsilon,d} := \Big\{ B \in \mathcal{R}_d \ \mid \ |B| \geq \varepsilon \Big\}.$$

**Definition.** *We say that* $\mathcal{N} \subset \mathcal{R}_d$ *is a* $\delta$*-net for* $\mathcal{B}_{\varepsilon,d}$ *if*

$$\forall\, B \in \mathcal{B}_{\varepsilon,d} \ \exists\, B_0 \in \mathcal{N} : \quad B_0 \subset B \quad \text{and} \quad |B_0| \geq (1-\delta)|B|.$$

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A variant of the following lemma using random points and the union bound was proved by Rudolf.

## Lemma (size of a net)

*Let* $d \geq 1$ *and* $\varepsilon, \delta \in (0,1)$. *Let* $\mathcal{N}$ *be a* $\delta$*-net for* $\mathcal{B}_{\varepsilon,d}$ *with* $|\mathcal{N}| \geq 3$. *Then with probability at least* $1 - 1/|\mathcal{N}|$

$$N = \left\lfloor \frac{3 \ln |\mathcal{N}|}{(1-\delta)\varepsilon} \right\rfloor.$$

*a random choice of N points satisfies the desire property.*

# Proof of the lemma.

Consider $N$ independent random points $X_1, ..., X_N$ uniformly drawn from $[0,1]^d$. It is enough to show that

$$\forall B \in \mathcal{N} \quad \text{with } |B| \geq v = (1-\delta)\varepsilon \;\; \exists j \leq N : \quad X_j \in B.$$

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Therefore, by the union bound,

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Thus, as far as

$$|\mathcal{N}| \exp(-vN) \leq 1$$

there exists a realization of $X_j$'s with the desired property.

# Construction of a net.

# Dispersion on the torus.

We consider periodic axis parallel boxes, that is, boxes of the form

$$\prod_{i=1}^{d} I_i(a_i, b_i), \quad a_i, b_i \in [0, 1],$$

where

$$I_i(a, b) := \begin{cases} (a_i, b_i), & \text{whenever } 0 \le a_i < b_i \le 1, \\ [0, 1] \setminus [b_i, a_i], & \text{whenever } 0 \le b_i < a_i \le 1. \end{cases}$$

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Combining recent results of M. Ullrich (**18**) and Rudolf (**18**)

$$\frac{d}{\varepsilon} \le \widetilde{N}(\varepsilon, d) \le \frac{8d}{\varepsilon} \left( \ln d + \ln \left( \frac{8}{\varepsilon} \right) \right).$$

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Note that the VC dimension of the set of periodic axis parallel boxes is of the order $d \ln d$, therefore the Blumer–Ehrenfeucht–Haussler–Warmuth result leads to

$$\widetilde{N}(\varepsilon, d) \le \frac{8d \ln d}{\varepsilon} \ln \left( \frac{8}{\varepsilon} \right)$$

— worse than the Rudolf bound.

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We improve the Rudolf upper bound in the case $\varepsilon \leq 1/d^C$.

## Theorem (bounds in the periodic case)

*Let* $d \geq 2$ *and* $\varepsilon \in (0, 1/2]$. *Then*

(*i*) $\widetilde{N}(\varepsilon, d) \leq \frac{C \ln d}{\varepsilon} \ln \left( \frac{1}{\varepsilon} \right)$, *provided that* $\varepsilon \leq \exp(-d)$,

(*ii*) $\widetilde{N}(\varepsilon, d) \leq \frac{C\, d\, \ln d}{\varepsilon}$, *provided that* $\varepsilon \geq \exp(-d)$.

*Moreover, the random choice of points uniformly distributed on* $[0, 1]^d$ *works.*