Space vectors forming rational angles

(a question of J.H. Conway)

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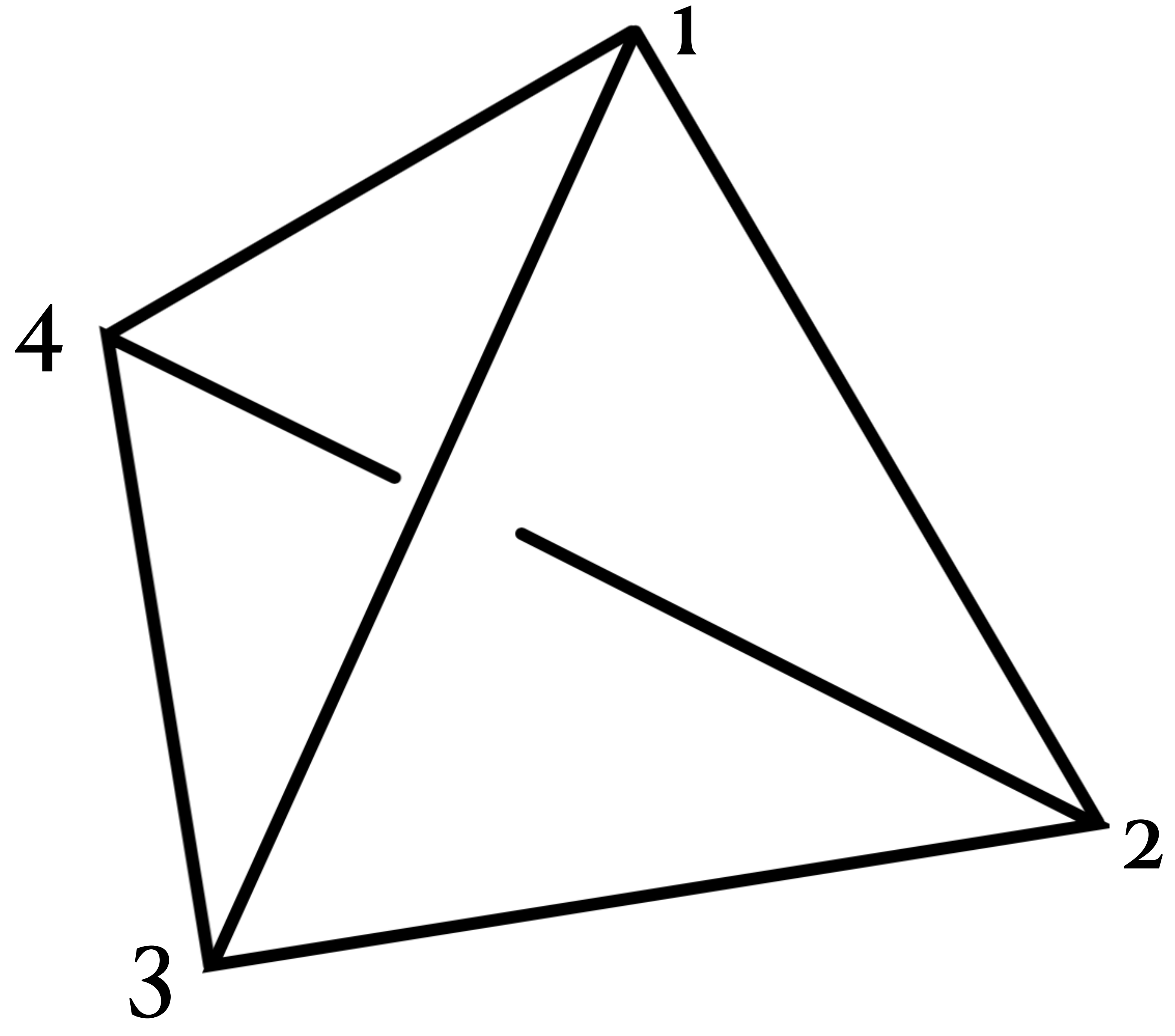
The main question

as it appears in the work of Conway and Jones (1976)

Classify all tetrahedra in $\mathbb{R}^3$ with "rational" angles, i.e. classify all 6-tuples of the form $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}) \in (\pi\mathbb{Q})^6$ that correspond to (non-degenerate) tetrahedra as shown below: an α_{ij} angle corresponds to the edge joining vertices i and j.



Another question

which is more ancient (and way harder!)

Classify all rectifiable tetrahedra in $\mathbb{R}^3$, i.e.

those that are scissors—congruent to a cube

Scissor congruence: two polytopes P and P' are scissors—congruent

if one can cut P into a finite number of tetrahedra and reassemble P'

(this definition is not very formal ...)

Scissor congruence

and its invariants

One (obvious) invariant is volume: we are cutting into a finite amount of pieces and reassembling, so that no Banach—Tarski paradox takes place

Are there other (independent) invariants?

Scissors congruence

and its invariants

The Dehn invariant of a polytope $P \subset \mathbb{R}^3$ is given by

$$D(P) = \sum_{edges\ e} length(e) \otimes_{\mathbb{Q}} angle(e) \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Z})$$

Theorem (Dehn $\Rightarrow$ in 1901, Sydler $\Leftarrow$ in 1964) :

Two polytopes P and P' in $\mathbb{R}^3$ are scissors–congruent

if and only if $vol(P) = vol(P')$ and $D(P) = D(P')$

(and, later on, Jessen gave a simpler proof ...)

The main result

(well, one of the two ...)

Theorem (KKPR, 2020): There are 2 parametric families of "rational" tetrahedra,

$$(\pi/2, \pi/2, \pi - 2t, \pi/3, t, t), \quad \pi/6 < t < \pi/2,$$

and

$$(5\pi/6 - t, \pi/6 + t, 2\pi/3 - t, 2\pi/3 - t, t, t), \quad \pi/6 < t \leq \pi/3,$$

together with 59 sporadic instances, up to geometric symmetries.

N	Dihedral angles as multiples of π/N
12 24	(3, 4, 3, 4, 6, 8) (5, 9, 6, 8, 13, 15)
12 24	(3, 6, 4, 6, 4, 6) (7, 11, 7, 13, 8, 12)
15	(3, 3, 3, 5, 10, 10), (2, 4, 4, 4, 10, 10), (3, 3, 4, 4, 9, 11)
15	(3, 3, 5, 5, 9, 9)
15	(5, 5, 5, 9, 6, 6), (3, 7, 6, 6, 7, 7), (4, 8, 5, 5, 7, 7)
21	(3, 9, 7, 7, 12, 12), (4, 10, 6, 6, 12, 12), (6, 6, 7, 7, 9, 15)
30 60	(6, 12, 10, 15, 10, 20), (4, 14, 10, 15, 12, 18) (8, 28, 19, 31, 25, 35), (12, 24, 15, 35, 25, 35),(13, 23, 15, 35, 24, 36), (13, 23, 19, 31, 20, 40)
30	(6, 18, 10, 10, 15, 15), (4, 16, 12, 12, 15, 15), (9, 21, 10, 10, 12, 12)
30 60	(6, 6, 10, 12, 15, 20) , (5, 7, 11, 11, 15, 20) (7, 17, 20, 24, 35, 35), (7, 17, 22, 22, 33, 37), (10, 14, 17, 27, 35, 35), (12, 12, 17, 27, 33, 37)
30 60	(6, 10, 10, 15, 12, 18), (5, 11, 10, 15, 13, 17) (10, 22, 21, 29, 25, 35), (11, 21, 19, 31, 26, 34), (11, 21, 21, 29, 24, 36), (12, 20, 19, 31, 25, 35)
30 60	(6, 10, 6, 10, 15, 24) (7, 25, 12, 20, 35, 43)
30 60	(6, 12, 6, 12, 15, 20) (12, 24, 13, 23, 29, 41)
30 60	(6, 12, 10, 10, 15, 18), (7, 13, 9, 9, 15, 18) (12, 24, 17, 23, 33, 33), (14, 26, 15, 21, 33, 33), (15, 21, 20, 20, 27, 39), (17, 23, 18, 18, 27, 39)
30 60	(6, 15, 6, 18, 10, 20), (6, 15, 7, 17, 9, 21) (9, 33, 14, 34, 21, 39), (9, 33, 15, 33, 20, 40), (11, 31, 12, 36, 21, 39), (11, 31, 15, 33, 18, 42)
30	(6, 15, 10, 15, 12, 15), (6, 15, 11, 14, 11, 16), (8, 13, 8, 17, 12, 15), (8, 13, 9, 18, 11, 14), (8, 17, 9, 12, 11, 16), (9, 12, 9, 18, 10, 15)
30 60	(10, 12, 10, 12, 15, 12) (19, 25, 20, 24, 29, 25)

The main result

(a more general statement)

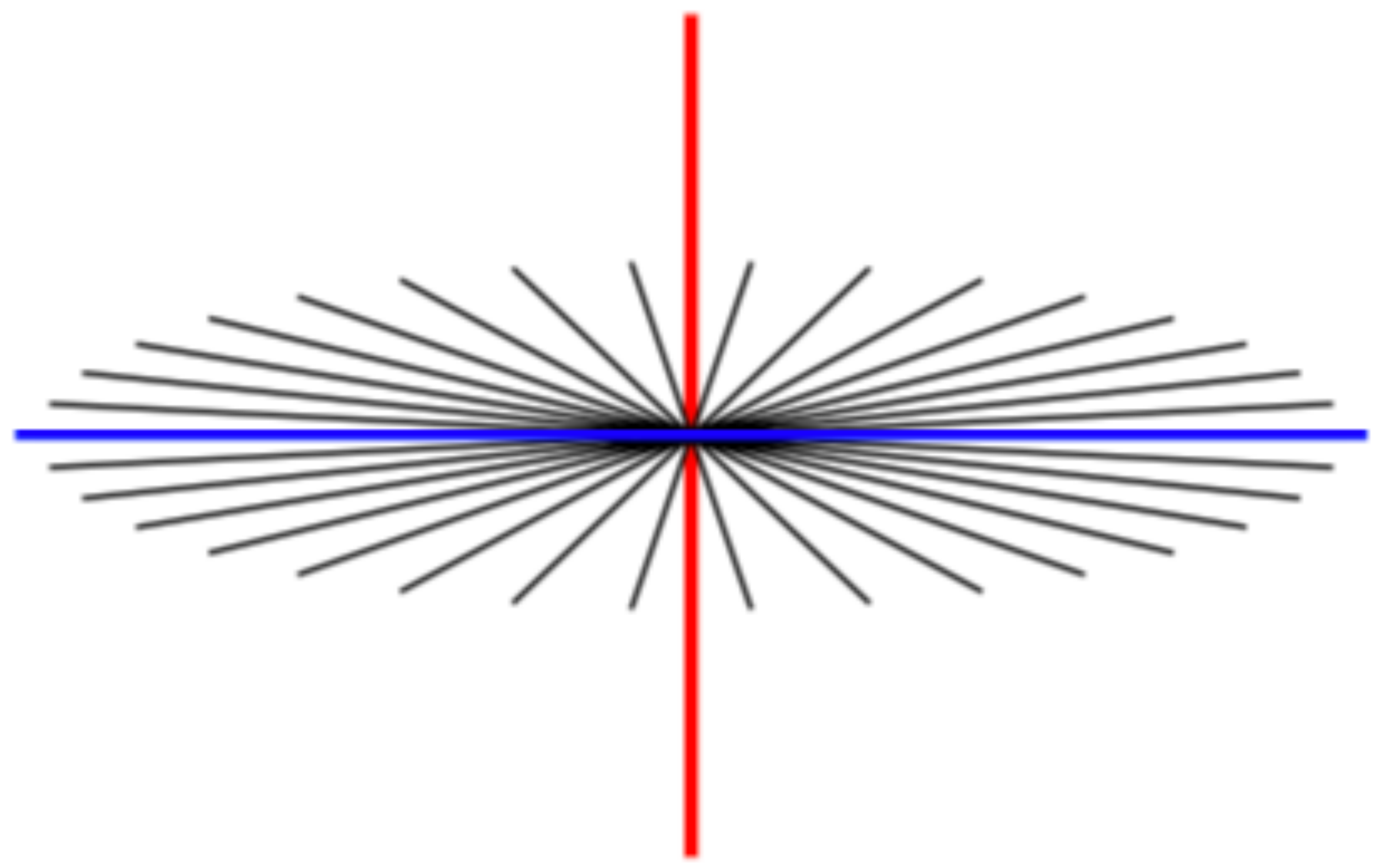
Theorem (KKPR, 2020): Up to geometric symmetries, there are finitely many maximal "rational" line configurations in $\mathbb{R}^3$ (i.e. line configurations with all angles between the lines $\in \pi\mathbb{Q}$).

n	# of maximal "rational" n—line configurations
$\aleph_0$	1
15	1
9	1
8	5
6	22 sporadic, 5 one-parameter families
5	29 sporadic, 2 one-parameter families
4	228 sporadic, 10 one-parameter and 2 two-parameter families
3	a single 3-parameter family

The main result

(some more comments)

The only infinite maximal configuration is the perpendicular one.

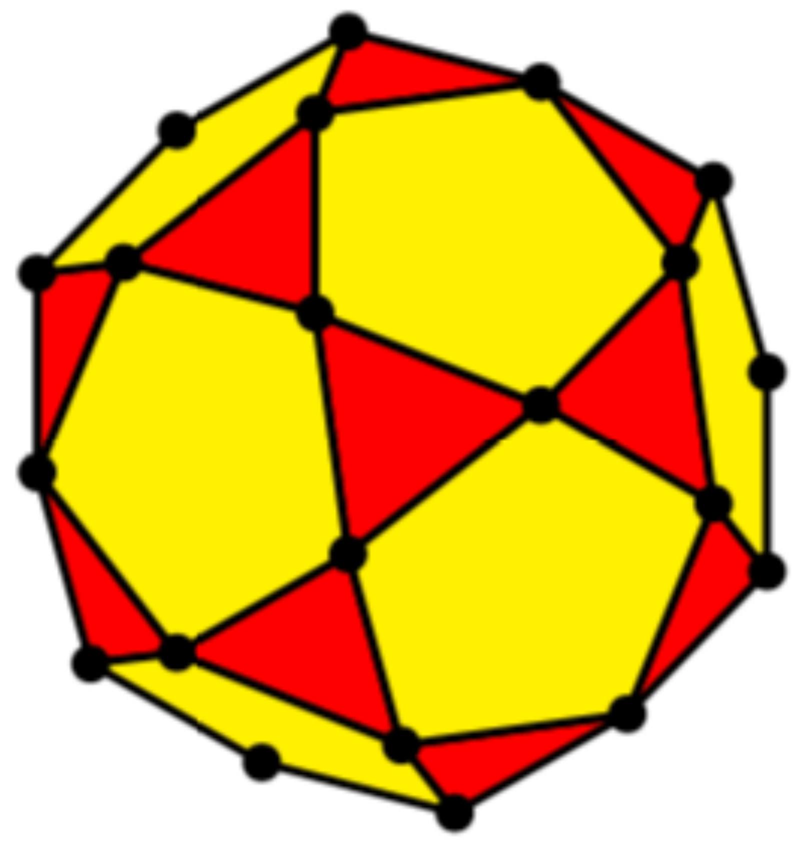


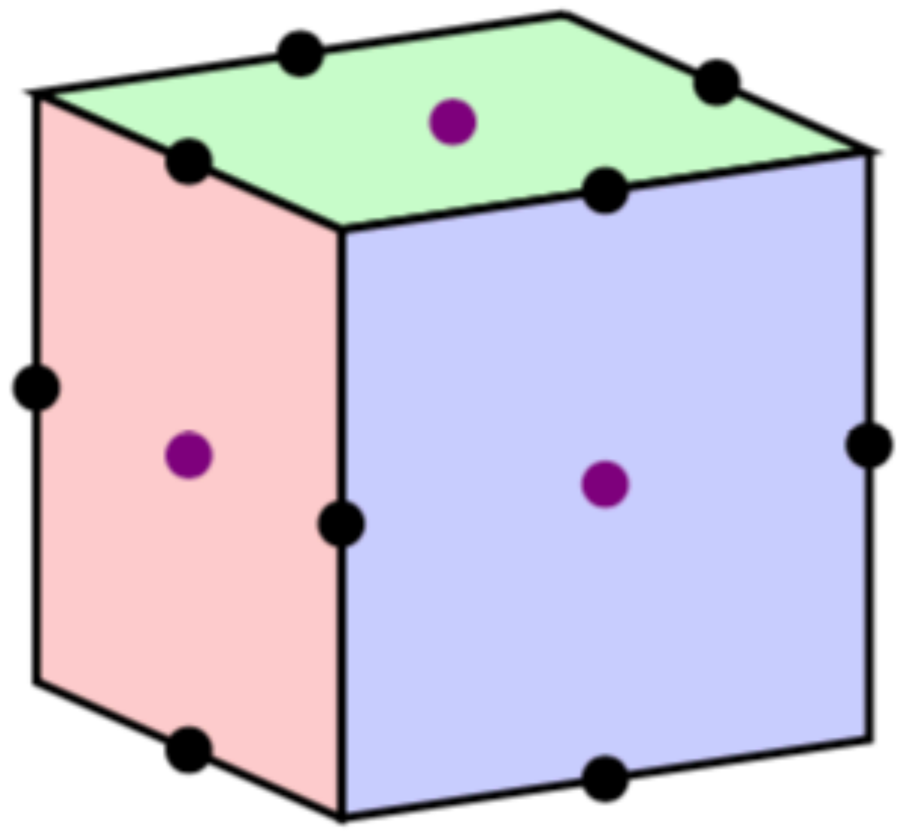
This configuration has to be infinite (for maximality).

The main result

(some more comments)

There are other two more remarkable ones:





the one coming from the icosidodecahedron (left) with 15 lines

and the one coming from the B_3 roots system (right) with 9 lines

Regge symmetry

and its properties

Theorem (Ponzano—Regge, 1960's): For any (non-degenerate) tetrahedron with dihedral angles $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ there exists a (non-degenerate) tetrahedron with dihedral angles $(\alpha_{12}, \alpha_{34}, \sigma - \alpha_{13}, \sigma - \alpha_{24}, \sigma - \alpha_{14}, \sigma - \alpha_{23})$, where $\sigma = \frac{1}{2}(\alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23})$.

This theorem was discovered by theoretical physicists studying quantum gravity.

Regge symmetry

and its properties

The Regge group $\mathfrak{R}$ is the symmetric group $\mathfrak{S}_4$ (geometric symmetries of the tetrahedron) enhanced by the Regge involution (from the previous slide). This is a finite group of order 144 that is isomorphic to $\mathfrak{S}_4 \times \mathfrak{S}_3$.

The Regge group $\mathfrak{R}$ preserves both the volume (Ponzano—Regge, 1960) and Dehn invariant (Roberts, 1999).

Regge symmetry

(back to our theorems)

The 1st family of tetrahedra $(\pi/2, \pi/2, \pi - 2t, \pi/3, t, t)$ was discovered by Hill (1895). The 2nd family $(5\pi/6 - t, \pi/6 + t, 2\pi/3 - t, 2\pi/3 - t, t, t)$ is produced by applying the Regge involution.

The entries in our table of sporadic tetrahedra are grouped with respect to the action of both $\mathfrak{S}_4$ (geometric symmetries) and $\mathfrak{R}$ (the Regge group).

Bigger symmetry groups

for line configurations, and other objects

For line configurations, however, we can also negate individual vectors that define line directions: thus, we use $\mathfrak{S}_4^{\pm} = \mathfrak{S}_4 \ltimes \{\pm 1\}^4$ instead (and also a "bigger" Regge group $\mathfrak{R}^{\pm}$).

The role of symmetries in our computations and proofs:

symmetry reduction greatly reduces the workload and runtime.

The biggest (finite) group we use is $W(D_6)$, the Weyl group of D_6 root system, of order 23040 (comes into play later on).

Computer—aided proofs

(a philosophical interlude)

Our main theorems have computer-aided proofs. We heavily use computations to find the answers, as well as to prove that they are correct (in particular, that we produce complete classifications).

This puts our results in the same category (take it with a grain of salt!) as the 4-colour theorem or the proof of Kepler's conjecture (a word to our defence: we don't claim the "significance" category, only the "methods of proofs" one!)

An outline of the proof

First, find the "rational" 4—line configurations (some of them will right away give the "rational" tetrahedra up to the transformation $\theta \to \pi - \theta =: \alpha$).

To this end, we find the 6—tuples of angles giving rise to symmetric matrices $\{\theta_{ij}\}_{i,j=1}^{4}$ that satisfy the Gram determinant equation

$$\det \begin{pmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{12} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{13} & \cos\theta_{23} & 1 & \cos\theta_{34} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & 1 \end{pmatrix} = 0.$$

An outline of the proof

We can rewrite it as a polynomial equation in the polynomial variables $z_{ij} = \exp(2i\theta_{ij})$:

$$\det \begin{pmatrix} 2 & z_{12} + z_{12}^{-1} & z_{13} + z_{13}^{-1} & z_{14} + z_{14}^{-1} \\ z_{12} + z_{12}^{-1} & 2 & z_{23} + z_{23}^{-1} & z_{24} + z_{24}^{-1} \\ z_{13} + z_{13}^{-1} & z_{23} + z_{23}^{-1} & 2 & z_{34} + z_{34}^{-1} \\ z_{14} + z_{14}^{-1} & z_{24} + z_{24}^{-1} & z_{34} + z_{34}^{-1} & 2 \end{pmatrix} = 0.$$

Above we have a Laurent polynomial in 6 variables: let's call it the Gram polynomial.

Solving it in the roots of unity will give us "rational" solutions to the initial Gram determinant equation.

Solving Laurent polynomials in the roots of unity

There are two methods that we can try to employ: classifying vanishing sums of roots of unity (known up to 12 roots, by the work of Mann, Włodarski, Conway and Jones, Poonen and Rubinstein), and using commutative algebra to obtain torsion closures of polynomial ideals (following the work of Ruppert, Beukers and Smyth, and Aliev and Smyth).

Solving Laurent polynomials in the roots of unity

Vanishing sums: let $\zeta_1 + \ldots + \zeta_n$ be a sum of roots of unity. Then if $n = 6$ the only the following cases hold:

- ζ_i's cancel in pairs: $\zeta \cdot 1 + \zeta \cdot e^{i\pi}$, for some $\zeta \in \mathbb{S}^1$ (i.e. up to rotation, for each pair);
- ζ_i's form two triples of the form $\zeta \cdot 1 + \zeta \cdot e^{2\pi/3} + \zeta \cdot e^{4\pi/3}$, for some $\zeta \in \mathbb{S}^1$, for each triple;
- as a set ζ_i's are $\{-e^{-2\pi i/3}, -e^{4\pi i/3}, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}\}$ (up to simultaneous rotation).

Note: vanishing sums are classified up to 12 roots of unity.

Solving Laurent polynomials in the roots of unity

Torsion closures: this method is base of the following observation + induction by the dimension of the corresponding algebraic variety and the number of variables.

If $f(x, y) = 0$, for x, y roots of unity, then (depending on the orders) x, y also solve one of the following polynomials: $f(-x, y)$, $f(x, -y)$, $f(-x, -y)$, $f(x^2, y^2), f(-x^2, y^2)$, $f(x^2, -y^2)$, or $f(-x^2, -y^2)$. Then we take the resultant to intersect the two resulting curves. The torsion closure of the ideal $I = \langle f(x, y) \rangle$ is obtain in this way after a finite number of steps.

Note: this method works well for $n = 2$ variables, but becomes infeasible for $n \geq 4$.

Solving Laurent polynomials in the roots of unity

The Gram polynomial is

$$-20+4\sum z_{12}^{\pm 1}z_{13}^{\pm 1}z_{23}^{\pm 1}-2\sum z_{12}^{\pm 2}-2\sum z_{12}^{\pm 1}z_{13}^{\pm 1}z_{24}^{\pm 1}z_{34}^{\pm 1}+\sum z_{12}^{\pm 2}z_{34}^{\pm 2}$$

where each sum ranges of the $\mathfrak{S}_4$—orbit of each monomial and each possible choice of signs. It enjoys a large symmetry group: $W(D_6)$ of order 23040.

There we have $1+4\cdot 2^3+6\cdot 2^1+3\cdot 2^4+3\cdot 2^2=105$ monomials and 6 variables.

Solving Laurent polynomials in roots of unity

An obvious complication: we have a polynomial with 105 monomial in 6 variables!

This is beyond the reach of both approaches outlined above

(105 is much bigger than 12, and 6 variables are about 2 variables too many!)

An outline of the proof

(reduce mod 2 and then come back)

Finally, we get a chance to unite both approaches by using the following observation:

reducing modulo 2 produces the polynomial

$$\sum z_{12}^{\pm 2} z_{34}^{\pm 2}$$

with 12 monomials and 6 variables.

By classifying vanishing sums of roots of unity mod 2 (which is similar to the classification of Poonen and Rubinstein), we obtain families of solutions with up to 3 free parameters (= variables). Then we can proceed to torsion closures!

An outline of the proof

(main steps)

- Start with a computation to find solutions with small denominators (up to 420). This gives a conjectural classification for sporadic instances. This is done by using C/C++ code. (Also SageMath to double-check)
- Classify all vanishing sums of up to 12 roots of unity and match them with the 12 monomials in the mod 2 reduced Gram polynomial. Then come back to the original Gram polynomial. (SageMath)
- Use $W(D_6)$ symmetry to reduce the number of equations down to a feasible amount. Solve them by finding torsion closures via the commutative algebra approach. This gives all parametric solutions. This also confirms that all sporadic ones have denominators ≤ 420. (SageMath)
- For the parametric solutions, use the geometry of convex polyhedra (in $\mathbb{R}^3$, since we have to deal with line configurations and tetrahedra, and the existence conditions, as well as in $\mathbb{R}^6$, since we have 6—tuples of angles describing our solutions!) in order to convert algebraic solutions into geometric solutions (for parametric families). (SageMath)

An outline of the proof

(main steps)

As soon as we are done classifying $4-$line configurations, we can attempt classifying all $n-$line configurations with $n \geq 5$ by using the following fact: an $n-$line configuration is realisable if and only if each of its $4-$line subconfigurations is realisable.

Thus we can try to repeatedly enhance each of the already found $4-$line configurations, and check if all their $4-$line subconfigurations belong to the list that we compiled (by using both sporadic and parametric solutions). This, however, requires an extensive tree search (here MAGMA is used).

We check up to $n = 16$ in order to make sure that there are no rational line configurations with more than 15 lines, except for the perpendicular one. Finally, only the maximal ones are listed.

Thank you !

(more details in arXiv:2011.14232)