A sharp threshold for Ramsey's theorem

Wojciech Samotij (Tel Aviv University)

joint work with

Ehud Friedgut, Eden Kuperwasser, and Mathias Schacht

MIPT Big Seminar

Theorem (Ramsey 1930). If $n \geq n(k,r)$, then every r-colouring of $E(K_n)$ yields a monochromatic K_k.

$$G \to (H)_r \equiv \forall c : E(G) \to \{1,\dots,r\}\ \exists i\ c^{-1}(i) \supseteq H$$

'G is Ramsey for H in r colours'

Corollary. If $n \geq n(H,r)$, then $K_n \to (H)_r$.

This talk. Replace K_n with $G_{n,p}$ and study the function

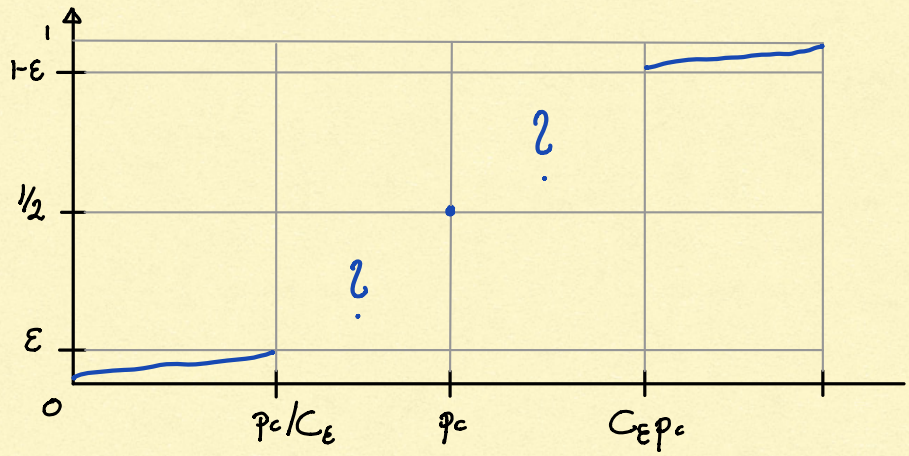
$$\mu(p) = \mu_{H,r,n}(p) := \mathbb{P}(G_{n,p} \to (H)_r)$$

$$\mu(p) = \mathbb{P}(G_{n,p} \to (H)_r)$$

Facts. • $\mu(0) = 0$ • $\mu(1) = 1$ • μ is increasing & continuous

$\emptyset \not\to (H)_r$ $\quad$ $K_n \to (H)_r$ $\quad$ $p \le p' \Rightarrow G_{n,p} \subseteq G_{n,p'}$ $\quad$ μ is a polynomial

Theorem (Bollobás–Thomason 1987): Let $p_c := \mu^{-1}(1/2)$. $\forall \varepsilon > 0\ \exists C_\varepsilon$



Definition. A function $\hat{p} : \mathbb{N} \to [0,1]$ is a **threshold** for a graph property $\mathcal{P} = (\mathcal{P}_n)$ if

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{P}_n) = \begin{cases} 1, & p \gg \hat{p} \\ 0, & p \ll \hat{p} \end{cases}$$

$p(n)/\hat{p}(n) \to \infty$

$p(n)/\hat{p}(n) \to 0$

Theorem (Bollobás-Thomason 1987). Every monotone and nontrivial graph property has a threshold.

Question. The property $G \to (H)_r$ is monotone & nontrivial. Where is the threshold?

Theorem (Rödl-Ruciński 1995). For all* H and $r \geq 2$, there are $c_0, c_1 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \to (H)_r) = \begin{cases} 1, & p \geq c_1 n^{-\frac{1}{m_2(H)}} \\ 0, & p \leq c_0 n^{-\frac{1}{m_2(H)}} \end{cases}$$

where $m_2(H) := \max\left\{ \frac{e_F - 1}{v_F - 2} : F \subseteq H,\ e_F \geq 2 \right\}$ (2-density).

* unless H is a forest of stars or ($H =$  & $r=2$)

(e.g.  )

$H = K_3$, $r = 2$: Frankl-Rödl (1986) & Łuczak-Ruciński-Voigt (1992)

Intuition behind the location of the threshold:

If $p = cn^{-\frac{1}{m_2(H)}}$ & $F \subseteq H$ satisfies $m_2(H) = \frac{e_F - 1}{v_F - 2}$, then

$$\mathbb{E}[\underbrace{\#\text{copies of } F \subseteq G_{n,p} \text{ containing } e}_{N(e,F)} \mid e \in G_{n,p}] = \Theta(c^{e_F - 1}).$$

0-statement: If $N(e,F) \leq r-1$ for some $e \in G_{n,p}$, then one can extend any F-free r-colouring to e.

1-statement: See Nenadov–Steger (2016).

Definition. A function $\hat{p} : \mathbb{N} \to [0,1]$ is a **sharp threshold** for a graph property $\mathcal{P} = (\mathcal{P}_n)$ if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(G_{n,p} \in \mathcal{P}_n) = \begin{cases} 1, & p \geq (1+\varepsilon)\hat{p} \\ 0, & p \leq (1-\varepsilon)\hat{p} \end{cases}$$

Examples:

- sharp th: connectivity, Hamiltonicity $\qquad \hat{p} = \log n / n$
- coarse (non-sharp) th: $\cdot \supseteq K_3$ $\qquad \hat{p} = 1/n$

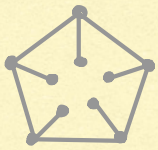
$$\mathbb{P}(G_{n,c/n} \not\supseteq K_3) = \exp(-c^3/6 + o(1))$$

Friedgut: sharp th ~ 'global' ppties & coarse th ~ 'local' ppties ?

Theorem (Friedgut 1999). If a graph ppty has a coarse th, then it correlates with a 'local' ppty.

Question: Is the ppty $G_{n,p} \to (H)_r$ 'local' or 'global'?

Theorem (Friedgut-Krivelevich 2000) $G_{n,p} \to (K_{1,k})_r$ and $G_{n,p} \to (\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet)_2$ have coarse th; for every other tree T & $r \geq 2$, $G_{n,p} \to (T)_r$ has a sharp th.

$K_{1,(k-1)r+1} \to (K_{1,k})_r$ &  $\longrightarrow (\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet)_2$

& both these graphs appear at/below $\Theta(n^{-\frac{1}{m_2(T)}}) = \Theta(n^{-1})$.

Theorem (Friedgut-Rödl-Ruciński-Tetali 2006). $G_{n,p} \to (K_3)_2$ has a sharp th. (65 pages)

Conjecture (FRRT). $G_{n,p} \to (H)_r$ has a sharp th if H = cycle.

Theorem (Schacht-Schulenburg 2018). $G_{n,p} \to (H)_2$ has a sharp th when H is strictly 2-balanced & 'nearly bipartite'.

$m_2(F) < m_2(H)$ for $F \subsetneq H$ $\qquad$ $\chi(H \setminus e) = 2$ for some $e \in H$

Friedgut-Hàn-Person-Schacht (2015) $(\mathbb{Z}_N)_p \to (k\text{-AP})_2$ (19 pp.)

All these results assume $r=2$.

Theorem (Friedgut–Kuperwasser–S.–Schacht 2020+).

If $H \in \{\text{clique}, \text{cycle}, \ldots\}$, then $G_{n,p} \to (H)_r$ has a sharp th for all $r \geq 2$.

strictly 2-balanced & two extra ppties

The remainder of this talk: proof outline for $H = K_3$ & $r = 3$.

Let $\mu(p) = \mathbb{P}(G_{n,p} \to (K_3)_3)$ and assume that th is coarse.

$\Rightarrow \exists \underset{\text{interval}}{I} \subseteq [c_0 n^{-1/2}, c_1 n^{-1/2}]$ $|I| = \Omega(n^{-1/2}) \wedge \forall p \in I$ $\Omega(1) \leq \mu(p) \leq 1 - \Omega(1)$

$\Rightarrow \exists p \in I$ $\mu'(p) = O(p^{-1})$

Thm (Friedgut 1999): $\exists \delta > 0$ & graph B with $O(1)$ edges s.t.:

- $\mathbb{P}(G_{n,p} \cup \varphi(B) \to (K_3)_3) \geq \mu(p) + \delta$

  a uniformly random (equiv. fixed) copy of B in K_n

- $\mathbb{P}(B \subseteq G_{n,p}) = \Omega(1)$ o/w one could take $B = K_m \to (K_3)_3$

Corollary. $\exists\, \varepsilon > 0$ & graph B with $O(1)$ edges s.t. for $Z \sim G_{n,p}$, with probability $\geq \varepsilon$:

(a) $\mathbb{P}(Z \cup G_{n,\varepsilon p} \to (K_3)_3 \mid Z) \leq 1/2$,

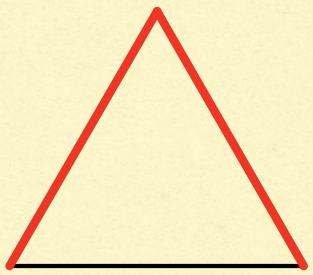
(b) $\mathbb{P}(Z \cup \varphi(B) \to (K_3)_3 \mid Z) \geq \varepsilon$.

random copy of B in $\mathcal{F}_n$

We fix one Z satisfying (a)&(b); by ——, we may ask Z avoid any 'bad' event $\mathcal{B}$ s.t. $\mathbb{P}(G_{n,p} \in \mathcal{B}) < \varepsilon$.

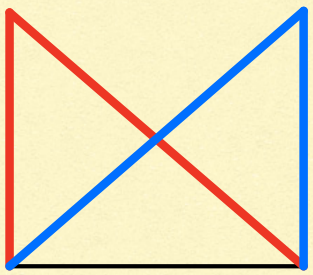
By (a), $Z \not\to (K_3)_3$, so $\exists$ proper colouring(s) $Z \to \{R, G, B\}$.

Fix some proper colouring $Z \to \{R, G, B\}$.



$e \in K_n$

R is unavailable at e



$e \in K_n$

R, B are unavailable at e

$\Downarrow$

G is forced at e

Observation. If a triangle T is forced to some colour, then the colouring cannot be extended to $Z \cup T$.

Proof outline:

① Show that every proper colouring of Z:

Ⓔ forces $\Omega(n^2)$ edges to some colour,

Ⓣ forces $\Omega(n^3)$ triangles to some colour.

② Bound $\mathbb{P}(G_{n,\varepsilon p}$ avoids all forced K_3s$)$ from above for every fixed proper colouring of Z.

③ Union bound over all colourings:

$$1/2 \overset{ⓐ}{\leq} \mathbb{P}(Z \cup G_{n,\varepsilon p} \not\to (K_3)_3) \leq \mathbb{P}\left(\begin{array}{l} G_{n,\varepsilon p} \text{ avoids all forced } K_3\text{s} \\ \text{for some proper colouring} \end{array}\right) = o(1) \quad ↯$$

(E) Every proper colouring of Z forces $\Omega(n^2)$ edges to same colour

Suppose only $o(n^2)$ edges are forced. Then

$$\mathbb{P}(\varphi(B) \text{ contains a forced edge}) \leq e(B) \cdot o(1) = o(1).$$

uniformly random copy of B in K_n

As K_3 is strictly 2-balanced & $e(Z) = O(n^{3/2})$

$$\mathbb{P}\left(Z \cap \varphi(B) \neq \emptyset \text{ or } \begin{array}{c} Z \cup \varphi(B) \text{ has a triangle } T \text{ s.t.} \\ 1 < e(T \cap \varphi(B)) < e(T) \end{array} \right) = o(1)$$

By (b), $\exists \varphi$ $\begin{cases} \bullet\ Z \cup \varphi(B) \to (K_3)_3 \\ \bullet\ \text{each } e \in \varphi(B) \text{ has } \geq 2 \text{ available colours} \\ \bullet\ K_3\text{s in } Z \cup \varphi(B): \end{cases}$ 

(i) $Z \cup \varphi(B) \to (K_3)_3$

$o(n^2)$ edges forced to a col.

$\Rightarrow$ (ii) each $e \in \varphi(B)$ has ≥ 2 available colours

(iii) K_3s in $Z \cup \varphi(B)$: 

Key property #1 : If $\mathbb{P}(B \subseteq G_{n,p}) = \Omega(1)$ (equiv. $m(B) \leq m_2(K_3)$),

then B is 2-choosable w.r.t. K_3s.

density

$\max\left\{\frac{e_F}{v_F} : \emptyset \neq F \subseteq H\right\}$

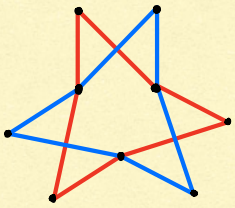
KP#1 $\overset{(ii)}{\Rightarrow}$ $\exists$ proper colouring of $\varphi(B)$ using available colours

$\overset{(iii)}{\Rightarrow}$ the proper colouring of Z can be extended to $\varphi(B)$

$\Rightarrow Z \cup \varphi(B) \not\to (K_3)_3$ $\lightning$ (i)

Ⓣ Every proper colouring of Z forces $\Omega(n^3)$ K_3s to some colour

Even though $\Omega(n^2)$ edges are forced to some colour, they could form a K_3-free graph...

Key property #2: If a partial $\{R, B\}$-colouring of K_n contains $\Omega(n^4)$ many  , then it must also contain $\Omega(n^9)$ many 

$$\text{KP\#2 \& Ⓔ} \xRightarrow[\text{second moment}]{\text{container lemma}} \text{Ⓣ}$$

② Bound $\mathbb{P}(G_{n,\varepsilon p}$ avoids all forced K_3s$)$

Janson's inequality $\Rightarrow \mathbb{P}(\ldots) \leq \exp(-c_\varepsilon n^2 p)$

③ Union bound over all colourings

$$1/2 \leq \mathbb{P}(Z \cup G_{n,\varepsilon p} \not\to (K_3)_3) \leq \#\text{colourings of } Z \times \exp(-c_\varepsilon n^2 p)$$

$$\#\text{colourings of } Z \leq 3^{e(Z)} \approx 3^{n^2 p/2}$$ ... hmmm ...

$$\mathbb{P}(\ldots) \geq \mathbb{P}(G_{n,\varepsilon p} = \emptyset) \approx \exp(-\varepsilon n^2 p)$$ ... oops ...

The union bound is too large!!!

Lemma. $\exists$ family $\mathcal{C}$ of $\exp(o(n^2p))$ partial colourings of Z:

(i) every proper col of Z extends some $c \in \mathcal{C}$,

(ii) every $c \in \mathcal{C}$ forces $\Omega(n^2)$ edges to some colour (and therefore also $\Omega(n^3)$ triangles).

Improved union bound:

$$1/2 \leq \mathbb{P}(Z \cup G_{n,\varepsilon p} \not\to (K_3)_3) \leq |\mathcal{C}| \times \exp(-c_\varepsilon n^2 p) = o(1). \quad \blacksquare$$

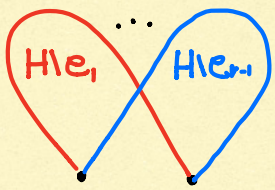
How to prove the lemma? Containers for a hypergraph on $Z \times \{R, G, B\}$ arising from $\varphi(B)$ s.t. $Z \cup \varphi(B) \to (K_3)_3$.

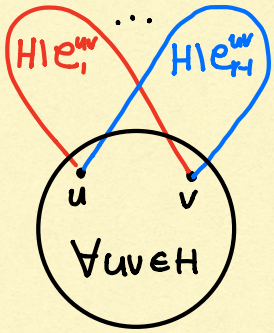
Sufficient conditions for $\mathrm{th}(G_{n,p} \to (H)_r)$ being sharp:

KP#0: H is strictly 2-balanced.

KP#1: If $m(B) \le m_2(H)$, then B is 2-choosable w.r.t. H from lists $\subseteq \{1,\dots,r\}$. RR $\Rightarrow$ KP#1 for $r=2$, all H

KP#2: For every partial $\{1,\dots,r-1\}$-col of K_n:





'easy' when H nearly bip. or H: clique

$$\# \geq \Omega\left(n^{2+(r-1)(v_H-2)}\right) \implies \# \geq \Omega\left(n^{v_H+(r-1)(v_H-2)}\right)$$