Generalizations of Steiner's porism and Soddy's hexlet

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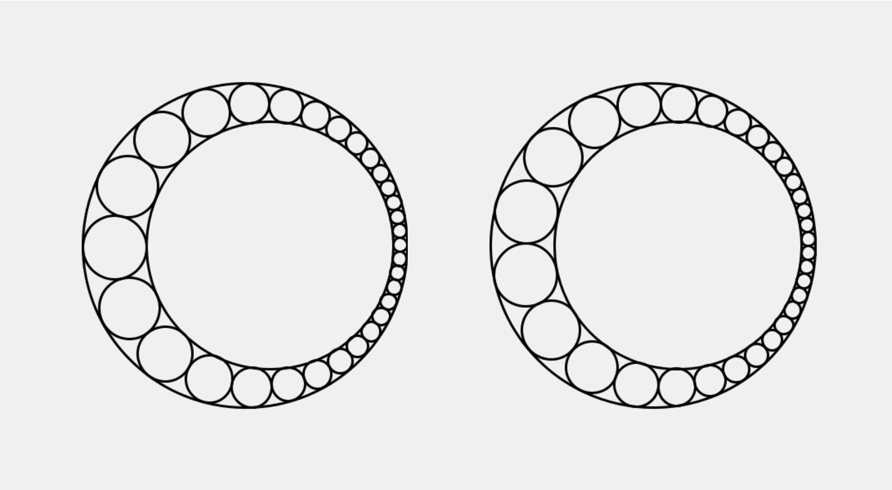
Steiner's porism

Suppose we have a chain of k circles all of which are tangent to two given non-intersecting circles S_1, S_2, and each circle in the chain is tangent to the previous and next circles in the chain. Then, any other circle C that is tangent to S_1 and S_2 along the same bisector is also part of a similar chain of k circles. This fact is known as *Steiner's porism*.

In other words, if a Steiner chain is formed from one starting circle, then a Steiner chain is formed from any other starting circle.

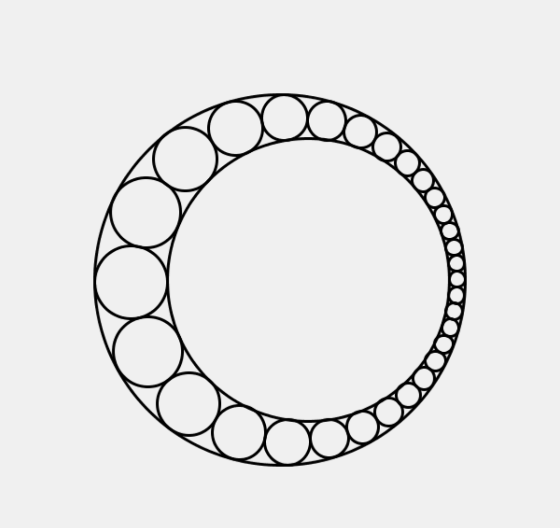
Equivalently, given two circles with one interior to the other, draw circles successively touching them and each other. If the last touches the first, this will also happen for any position of the first circle.

Steiner's porism

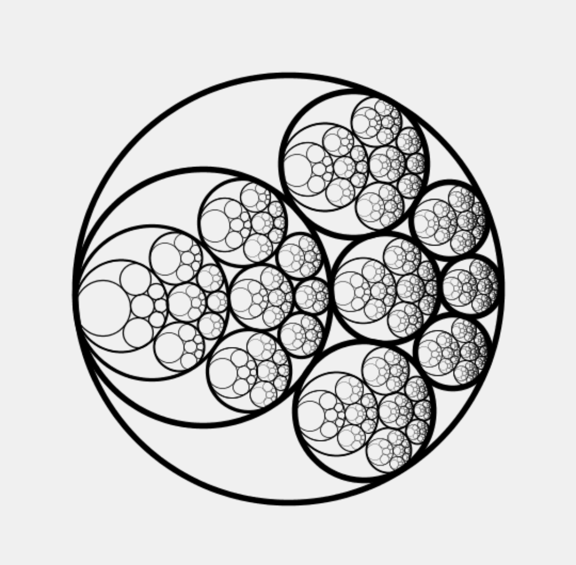


Steiner's chain

Gábor Damásdi



Steiner's chain



Jacob Steiner

Jakob Steiner (1796 – 1863) was a Swiss mathematician who was professor and chair of geometry that was founded for him at Berlin (1834-1863).

Steiner's mathematical work was mainly confined to geometry. His investigations are distinguished by their great generality, by the fertility of his resources, and by the rigour in his proofs. He has been considered the greatest pure geometer since *Apollonius of Perga*.

Porism

A *porism* is a mathematical proposition or corollary. In particular, the term porism has been used to refer to a direct result of a proof, analogous to how a corollary refers to a direct result of a theorem. In modern usage, a porism is a relation that holds for an infinite range of values but only if a certain condition is assumed, for example *Steiner's porism.* The term originates from three books of Euclid with porism, that have been lost. Note that a proposition may not have been proven, so a porism may not be a theorem, or for that matter, it may not be true.

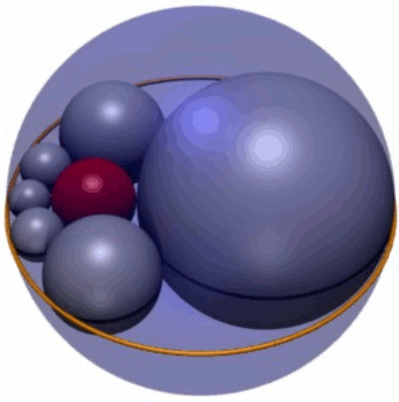
Soddy's hexlet

Soddy's hexlet is a chain of six spheres each of which is tangent to both of its neighbors and also to three mutually tangent given spheres.

According to a theorem published by Frederick Soddy in 1937:
It is always possible to find a hexlet for any choice of mutually tangent spheres A, B and C.

Moreover, there is an infinite family of hexlets related by rotation and scaling of the hexlet spheres.

Soddy's hexlet



Frederic Soddy

Frederick Soddy (1877 – 1956) was an English radiochemist who explained, with Ernest Rutherford, that radioactivity is due to the transmutation of elements, now known to involve nuclear reactions. He also proved the existence of isotopes of certain radioactive elements. He received the 1921 Nobel Prize in chemistry.

Soddy rediscovered the Descartes' theorem in 1936 and published it as a poem, "The Kiss Precise", quoted at Problem of Apollonius. The kissing circles in this problem are sometimes known as Soddy circles.

Hexlet's paper:
Soddy, Frederick. "The bowl of integers and the hexlet", *Nature,* London, **139** (1937), 77–79

Defining Inversions

An *inversion* is a transformation of the plane that has many common properties with the symmetry in a line. It can also be called *"symmetry in a circle"*, or circle inversion with respect to a given circle. Inversions preserve angles and map generalized circles into generalized circles, where a generalized circle means either a circle or a line (a circle with infinite radius). Many difficult problems in geometry become much more tractable when an inversion is applied.

The concept of an inversion can be generalized to higher dimensional spaces.

Defining Inversions

Definition

An inversion with respect to a circle of center O and radius r is a transformation for which every point P in the plane is mapped to the point P^* lying on the ray OP and such that

$$OP^* = \frac{r^2}{OP}.$$

We denote an inversion in a circle ω by I_ω, and by $G_{(O,k)}$ the homothety with center at a point O and coefficient k. When $P = O$ we need to divide by zero. Thus we consider an inversion as a selfmap of the extended plane $\mathbb{R}^2 \cup \infty$ (the plane with one additional point at infinity), and we see that $I_\omega(O) = \infty$ and $I_\omega(\infty) = O$.

Some Important Properties of Inversions

An important property of inversions is that they preserve the set of spheres in $\mathbb{R}^n$. More precisely:

Theorem (Image Under Inversion)

Let ω be a sphere centered at O and with radius r. Then images of spheres and planes under the inversion with respect to the sphere ω are described as follows:
1. The image of a sphere passing through the point O is a plane. Conversely, the image of a plane (not passing through the point O) is a sphere passing through the center O.
2. The image of any sphere not passing through the center O of ω is a sphere.

Steiner's porism

Theorem

Suppose that two nonintersecting circles ω_1 and ω_2 have the property that one can fit a "chain" of n circles between them, each tangent to the next. Then one can do this starting with any circle tangent to both given circles.

Proof

It can be shown that there exists an inversion with following property: images ω_1^* and ω_2^* of ω_1 and ω_2 under this inversion are concentric circles.

Suppose we have a "chain" between ω_1 and ω_2. Therefore we have a chain between ω_1^* and ω_2^*. But clearly we can rotate this "chain" between ω_1^* and ω_2^*. Under inversion with respect to ω this "rotated chain" will be mapped to a "chain" between ω_1 and ω_2.

Inversion T

Let S_1 and S_2 be spheres in $\mathbb{R}^n$. Consider two cases:
(i) S_1 and S_2 are tangent;
(ii) S_1 and S_2 do not have common points.

In case (i) let O be the contact point of these spheres and if we apply the sphere inversion T with center O and an arbitrary radius ρ, then S_1 and S_2 become two parallel hyperplanes S_1' and S_2'.
In case (ii) we can use the famous theorem: *There is T that invert S_1 and S_2 into a pair of concentric spheres S_1' and S_2'.*

Lemma

The radius r_T of $S' = T(S)$ is the same for all spheres S that are tangent to S_1 and S_2.

Spherical codes

Let X be a set of points in a unit sphere $\mathbb{S}^{d-1}$. We say that X is a *spherical* ψ*–code* if the angular distance between distinct points in X is at least ψ.

Denote by $A(d, \psi)$ the maximal size of a ψ–code in $\mathbb{S}^{d-1}$. Note that $A(d, \pi/3) = k(d)$, where by $k(d)$ we denote the *kissing number*, i.e. the maximum number of non–overlapping unit spheres in $\mathbb{R}^d$ that can be arranged so that all of them touch one (central) unit sphere.

$\mathcal{F}$–kissing arrangements and spherical codes

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$, $2 \leq m < n+2$, be a family of m spheres in $\mathbb{R}^n$ such that S_1 and S_2 are non–intersecting or tangent spheres.
We say that a set $\mathcal{C}$ of spheres in $\mathbb{R}^n$ is an $\mathcal{F}$–kissing arrangement if
(1) each sphere from $\mathcal{C}$ is tangent all spheres from $\mathcal{F}$,
(2) any two distinct spheres from $\mathcal{C}$ are non–intersecting.

Theorem

For a given $\mathcal{F}$ the inversion T defines a one–to–one correspondence between $\mathcal{F}$–kissing arrangements and spherical $\psi_{\mathcal{F}}$–codes in $\mathbb{S}^{d-1}$, where $d = n + 2 - m$ and $\psi_{\mathcal{F}} \in [0, \infty]$ is uniquely defined by $\mathcal{F}$.

Analogs of Steiner's porism

Theorem

Let $\mathcal{F} = \{S_1, \ldots, S_m\}$, $2 \leq m < n+2$, be a family of m spheres in $\mathbb{R}^n$ such that S_1 and S_2 are non–intersecting spheres. If a Steiner packing is formed from one starting sphere, then a Steiner packing is formed from any other starting packing.

Steiner's packings

Proposition

If for a family $\mathcal{F}$ there exist a simplicial $\mathcal{F}$–kissing arrangement then we have one of the following cases

1. $d = 2$, $\psi_{\mathcal{F}} = 2\pi/k$, $k \geq 3$, *and* $P_{\mathcal{F}}$ *is a regular polygon with* k *vertices.*
2. $\psi_{\mathcal{F}} = \arccos(-1/d)$ *and* $P_{\mathcal{F}}$ *is a regular* d*–simplex with any* $d \geq 2$.
3. $\psi_{\mathcal{F}} = \pi/2$ *and* $P_{\mathcal{F}}$ *is a regular* d*–crosspolytope with any* $d \geq 2$.
4. $d = 3$, $\psi_{\mathcal{F}} = \arccos(1/\sqrt{5})$ *and* $P_{\mathcal{F}}$ *is a regular icosahedron.*
5. $d = 4$, $\psi_{\mathcal{F}} = \pi/5$ *and* $P_{\mathcal{F}}$ *is a regular 600–cell.*

Analogs of Soddy's hexlet

Let $m \geq 3$. Denote

$$\psi_m := \arccos\left(\frac{1}{m-1}\right).$$

Theorem

Let $3 \leq m < n+2$. Let X be a spherical ψ_m–codes in $\mathbb{S}^{d-1}$, where $d := n+2-m$. Then for any family $\mathcal{F}$ of m mutually tangent spheres in $\mathbb{R}^n$ there is an $\mathcal{F}$–kissing arrangement that is correspondent to X.

Analogs of Soddy's hexlet. Examples.

If $m = 3$, then $d = n - 1$, $\psi_3 = \pi/3$ and $A(n-1, \pi/3) = k(n-1)$. Consider *kissing arrangements* (maximum $\pi/3$–codes) in $\mathbb{S}^{d-1}$.

$n = 3$. We have $k(2) = 6$. Then a maximum $\pi/3$–code in $\mathbb{S}^1$ is a regular *hexagon.* The corresponding $\mathcal{F}$-sphere arrangement is a Soddy's hexlet.

$n = 4$. We have $k(3) = 12$. In this dimension there are infinitely many non–isometric kissing arrangements. Perhaps, the *cuboctahedron* can be a good analog of Soddy's hexlet in four dimensions.

$n = 5$. In four dimensions the kissing number is 24 and the best known kissing arrangement is a regular 24–*cell.* So in dimension five a nice analog of Soddy's hexlet is the 24–cell.

THANK YOU