Euclidean and spherical representation of graphs

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Abstract

Any graph G can be embedded in a Euclidean space as a two–distance set with the (minimum) distance equals a if the vertices are adjacent and distances equal b otherwise. The Euclidean representation number of G is the smallest dimension in which G is representable. In this talk we consider spherical and J–spherical representation numbers of G. We give exact formulas for these numbers using multiplicities of polynomials that are defined by the Caley–Menger determinant. We show that using W. Kuperberg's theorem the representation numbers can be found explicitly for the join of graphs.

Two–distance sets

A set S in Euclidean space $\mathbb{R}^n$ is called a *two-distance set*, if there are two distances a and b, and the distances between pairs of points of S are either a or b.

If a two-distance set S lies in the unit sphere $\mathbb{S}^{n-1}$, then S is called *spherical two-distance set.*

Euclidean representation of graphs

Let G be a graph on n vertices. Consider a *Euclidean representation of* G in $\mathbb{R}^d$ as a two distance set. In other words, there are two positive real numbers a and b with $b \geq a > 0$ and an embedding f of the vertex set of G into $\mathbb{R}^d$ such that

$$\operatorname{dist}(f(u), f(v)) := \begin{cases} a & \text{if } uv \text{ is an edge of } G \\ b & \text{otherwise} \end{cases}$$

We will call the smallest d such that G is representable in $\mathbb{R}^d$ the *Euclidean representation number* of G and denote it $\dim_2^{\mathrm{E}}(G)$.

Euclidean representation number of graphs

A complete graph K_n represents the edges of a regular $(n-1)$–simplex. So we have $\dim_2^{\mathrm{E}}(\mathrm{K}_n) = n - 1$. That implies

$$\dim_2^{\mathrm{E}}(G) \leq n - 1$$

for any graph G on n vertices.

Since for a two–distance set of cardinality n in $\mathbb{R}^d$

$$n \leq \frac{(d+1)(d+2)}{2}.$$

we have

$$\dim_2^{\mathrm{E}}(G) \geq \frac{\sqrt{8n+1}-3}{2}.$$

Einhorn and Schoenberg work

Einhorn and Schoenberg (ES66) proved that

Theorem

Let G be a simple graph on n vertices. Then $\dim_2^{\mathrm{E}}(G) = n - 1$ if and only if G is a disjoint union of cliques.

Einhorn and Schoenberg work on two–distance sets (1966)

Denote by Σ_n the number of all two–distance sets with n vertices in $\mathbb{R}^{n-2}$. Then

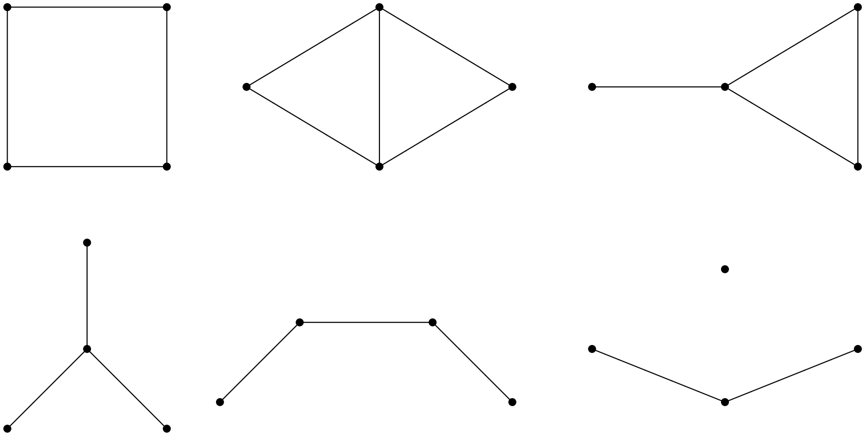
$$\Sigma_n = \Gamma_n - p(n),$$

where Γ_n is the number of all simple undirected graphs and $p(n)$ is the number of unrestricted partitions of n.

$$|\Gamma_4| = 11, \qquad |\Gamma_5| = 34, \qquad |\Gamma_6| = 156, \qquad |\Gamma_7| = 1044, \ldots$$

$$p(4) = 5, \qquad p(5) = 7, \qquad p(6) = 11\ , \qquad p(7) = 15, \ldots$$

$$|\Sigma_4| = 6, \qquad |\Sigma_5| = 27, \qquad |\Sigma_6| = 145, \qquad |\Sigma_7| = 1029, \ldots$$



Let $S = \{p_1, \dots, p_n\}$ in $\mathbb{R}^{n-1}$. Denote $d_{ij} := \mathrm{dist}(p_i, p_j)$.
Consider the Cayley–Menger determinant

$$C_S := \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{12}^2 & \dots & d_{1n}^2 \\ 1 & d_{21}^2 & 0 & \dots & d_{2n}^2 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & d_{n1}^2 & d_{n2}^2 & \dots & 0 \end{vmatrix}$$

Let S be a two-distance set with $a = 1$ and $b > 1$. Then for $i \neq j$,

$$d_{ij}^2 = 1 \text{ or } d_{ij}^2 = b^2$$

C_S is a polynomial in $t = b^2$.

Denote this polynomial by $C(t)$.

$$V_{n-1}^2(S) = \frac{(-1)^n\, C_s}{2^{n-1}\,((n-1)!)^2}$$

Actually, Einhorn and Schoenberg considered the discriminating polynomial $D(t)$ that can be defined through the Gram determinant. It is known that

$$C(t) = (-1)^n D(t)$$

Let G be a simple graph. Then

$$C_G(t) := C(t)$$

is uniquely defined by G.

Suppose there is a solution $t > 1$ of $C_G(t) = 0$.

Definition

Denote by τ_1 the smallest root t of C_G such that $t > 1$.

$\mu(G)$ denote the multiplicity of the root τ_1.

If for all roots t of C_G we have $t \leq 1$, then we assume that $\mu(G) := 0$.

The graph complement of G

If $\mu(G) > 0$, then $\tau_0(G) := 1/\tau_1(G)$ is a root of $C_{\bar{G}}(t)$ and $\tau_1(\bar{G}) = 1/\tau_0(G)$. Note that there are no more roots of $C_G(t)$ on the interval $[\tau_0(G), \tau_1(G)]$.

$C_{\bar{G}}(t)$ is the reciprocal polynomial of $C_G(t)$, i.e.

$$C_{\bar{G}}(t) = t^k C_G(1/t), \quad k = \deg C_G(t).$$

The Einhorn–Schoenberg theorem

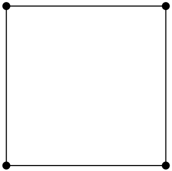
Theorem

Let G be a simple graph on n vertices. Then

$$\dim_2^{\mathrm{E}}(G) = n - \mu(G) - 1$$

If $\mu(G) > 0$, then a minimal Euclidean representation of G is uniquely define up to isometry.

$$C_1(t) = t^2(2 - t), \quad C_2(t) = t(3 - t), \quad C_3(t) = -t^2 + 4t - 1$$



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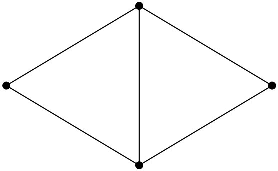

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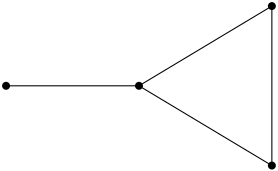

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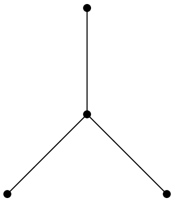
$$C_4(t) = t^2(3-t), \quad C_5(t) = (t+1)(3t-t^2-1), \quad C_6(t) = -t^2+4t-1$$



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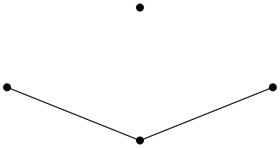

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$G = K_{2,\dots,2}$

Theorem

Let G be a complete m–partite graph $K_{2,\dots,2}$. Then $\dim_2^{\mathrm{E}}(G) = m$ and a minimal Euclidean representation of G is a regular cross–polytope.

Proof.

We have $n = 2m$ and

$$C_G(t) = 2m\, t^m (2-t)^{m-1}.$$

Then $\tau_1 = 2$ and $\mu(G) = m - 1$. Thus, $\dim_2^{\mathrm{E}}(K_{2,\dots,2}) = m$. □

V. Alexandrov (2016)

$G = K_{2,\dots,2}$: geometric proof

Lemma

Let for sets X_1 and X_2 in $\mathbb{R}^d$ there is $a > 0$ such that $\mathrm{dist}(p_1, p_2) = a$ for all $p_1 \in X_1$, $p_2 \in X_2$.
Then both X_i are spherical sets and the affine spans $\mathrm{aff}(X_i)$ in $\mathbb{R}^d$ are orthogonal each other.

Let $S := f(V(G))$ in $\mathbb{R}^d$. Then $\mathbb{R}^d$ can be split into the orthogonal product $\prod_{i=1}^m L_i$ of lines such that for $S_i := S \cap L_i$ we have $|S_i| = 2$. Thus, $d = m$ and S is a regular cross–polytope.

Spherical representations of graphs

Let f be a Euclidean representation of a graph G on n vertices in $\mathbb{R}^d$ as a two distance set. We say that f is a *spherical representation of* G if the image $f(G)$ lies on a $(d-1)$–sphere in $\mathbb{R}^d$. We will call the smallest d such that G is spherically representable in $\mathbb{R}^d$ the *spherical representation number* of G and denote it $\dim_2^{\mathrm{S}}(G)$.

Nozaki and Shinohara (2012) using Roy's results (2010) give a necessary and sufficient condition of a Euclidean representation of a graph G to be spherical.

We define a polynomial $M_G(t)$ and show that a Euclidean representation is spherical if and only if the multiplicity of $\tau_1(G)$ is the same for $C_G(t)$ and $M_G(t)$

Spherical representations of graphs

Let $S = \{p_1, \ldots, p_n\}$ be a set in $\mathbb{R}^{n-1}$. As above $d_{ij} := \mathrm{dist}(p_i, p_j)$. Let

$$M_S := \begin{vmatrix} 0 & d_{12}^2 & \ldots & d_{1n}^2 \\ d_{21}^2 & 0 & \ldots & d_{2n}^2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ d_{n1}^2 & d_{n2}^2 & \ldots & 0 \end{vmatrix}$$

The circumradius of a simplex

It is well known, that if the points in S form a simplex of dimension $(n-1)$, the radius R of the sphere circumscribed around this simplex is given by

$$R^2 = -\frac{1}{2}\frac{M_S}{C_S}.$$

Spherical representations of graphs

For a given graph G we denote by $M_G(t)$ the polynomial in $t = b^2$ that defined by M_S. Let

$$F_G(t) := -\frac{1}{2}\frac{M_G(t)}{C_G(t)}.$$

If G is a graph with $\mu(G) > 0$ and $F_G(\tau_1) < \infty$, then denote $\mathcal{R}(G) := \sqrt{F_G(\tau_1)}$. Otherwise, put $\mathcal{R}(G) := \infty$.

We will call $\mathcal{R}(G)$ the circumradius of G.

Spherical representations of graphs

Theorem

Let G be a graph on n vertices with $\mathcal{R}(G) < \infty$. Then $\dim_2^{\mathbb{S}}(G) = n - \mu(G) - 1$, *otherwise* $\dim_2^{\mathbb{S}}(G) = n - 1$.

The circumradius of a graph

Theorem

$\mathcal{R}(G) \geq 1/\sqrt{2}$.

It is not clear what is the range of $\mathcal{R}(G)$? If $\mathcal{R}(G) < \infty$, then for a fixed n there are only finitely many cases. Thus the range is a countable set.

Open question. *Suppose $\mathcal{R}(G) < \infty$. What is the upper bound of $\mathcal{R}(G)$? Can $\mathcal{R}(G)$ be greater than 1?*

J–spherical representation of graphs

We have $\mathcal{R}(G) \geq 1/\sqrt{2}$. Now consider the boundary case $\mathcal{R}(G) = 1/\sqrt{2}$.

Definition

Let f be a spherical representation of a graph G in $\mathbb{R}^d$ as a two distance set. We say that f is a J–spherical representation of G if the image $f(G)$ lies in the unit sphere $\mathbb{S}^{d-1}$ and the first (minimum) distance $a = \sqrt{2}$.

Theorem

For any graph $G \neq K_n$ there is a unique (up to isometry) J–spherical representation.

J–spherical representation of graphs

The uniqueness of a J–spherical representation of $G \neq K_n$ shows that the following definition is correct.

Definition

$\dim_2^{\mathrm{J}}(G)$ = *J–spherical representation dimension*
$b_*(G)$ = *the second distance of this representation.*

If G is the pentagon, then $\dim_2^{\mathrm{S}}(G) = 2 < \dim_2^{\mathrm{J}}(G) = 4$.

Theorem

Let $G \neq K_n$ be a graph on n vertices. If $\mathcal{R}(G) = 1/\sqrt{2}$, then

$$\dim_2^{\mathrm{J}}(G) = n - \mu(G) - 1, \text{ otherwise } \dim_2^{\mathrm{J}}(G) = n - 1.$$

W. Kuperberg's theorem

Rankin (1955) proved that if S is a set of $d+k$, $k \geq 2$, points in the unit sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, then two of the points in S are at a distance of at most $\sqrt{2}$ from each other. Wlodzimierz Kuperberg (2007) extended this result and proved that:

Theorem

Let d and k be integers such that $2 \leq k \leq d$. If S is a $(d+k)$–point subset of the unit d–ball such that the minimum distance between points is at least $\sqrt{2}$, then: (1) every point of S lies on the boundary of the ball, and (2) $\mathbb{R}^d$ can be split into the orthogonal product $\prod_{i=1}^{k} L_i$ nondegenerate linear subspaces so that for $S_i := S \cap L_i$, $d_i := \dim L_i$ we have $|S_i| = d_i + 1$ and $\mathrm{rank}(S_i) = d_i$ $(i = 1, 2, ..., k)$.

W. Kuperberg's theorem

Definition

The join $X * Y$ of two sets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ is formed in the following manner. Embed X in the m–dimensional linear subspace of $\mathbb{R}^{m+n}$ as

$$\{(x_1, \ldots, x_m, 0, \ldots, 0) : x = (x_1, \ldots, x_m) \in X\}$$

and embed Y as

$$\{(0, \ldots, 0, y_1, \ldots, y_n) : y \in Y\}.$$

Geometrically the join corresponds to putting the two sets X and Y in orthogonal linear subspaces of $\mathbb{R}^{m+n}$. So Kuperberg's theorem implies that $S = S_1 * \ldots * S_k$.

W. Kuperberg's theorem

Kuperberg's theorem can be slightly extended. What is a join–indecomposable spherical set? There are just two types.

Type I: $S \subset \mathbb{S}^{d-1}$, $|S| = d+1$, $\mathrm{rank}(S) = d$ and the center O of $\mathbb{S}^{d-1}$ lies in the interior of $\mathrm{conv}(S)$.

Type II: $S \subset \mathbb{S}^{d-1}$, $|S| = d$, $\mathrm{rank}(S) = d-1$ and $O \notin \mathrm{aff}(S)$.

Theorem

Let S be as in the Theorem. Then $S = S_1 * \ldots * S_m$, where S_i, $i = 1, \ldots, k$ are of Type I and all other S_i are of Type II.

Join of spherical two–distance sets

Definition

We say that a two–distance set S in $\mathbb{R}^d$ is a J–spherical two–distance set (JSTD) if S lies in the unit sphere centered at the origin 0 and $a = \sqrt{2}$. For this S the second distance b will be denoted $b(S)$.

Theorem

Let S_1 and S_2 be JSTD sets in $\mathbb{R}^d$. Then $S := S_1 \cup S_2$ is a JSTD set and $S = S_1 * S_2$ if and only if
(1) $\mathrm{dist}(p_1, p_2) = \sqrt{2}$ *for all points* $p_1 \in S_1$, $p_2 \in S_2$*;*
(2) $b(S_1) = b(S_2)$*;*
(3) $\mathrm{rank}(S \cup 0) = \mathrm{rank}(S_1 \cup 0) + \mathrm{rank}(S_2 \cup 0)$.

Representation numbers of the join of graphs

Recall that the *join* $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Representation numbers of the join of graphs

Definition

We say that G on n vertices is J–simple if $\dim_2^{\mathrm{J}}(G) = n - 1$.

Theorem

Let $G := G_1 + \ldots + G_m$. Suppose all G_i are J–simple and

$$b_*(G_1) = \ldots = b_*(G_k) < b_*(G_{k+1}) \leq \ldots \leq b_*(G_m).$$

Then

$$\dim_2^{\mathrm{J}}(G) = \dim_2^{\mathrm{S}}(G) = n - k, \;\; \dim_2^{\mathrm{E}}(G) = n - \max(k, 2),$$

where n denote the number of vertices of G.

Representation numbers of complete multipartite graphs

Corollary

Let G be a complete multipartite graph $K_{n_1 \ldots n_m}$. Suppose

$$n_1 = \ldots = n_k > n_{k+1} \geq \ldots \geq n_m.$$

Let $n := n_1 + \ldots + n_m$. Then

1. *If $k = 1$, then $\dim_2^{\mathrm{E}}(G) = n - 2$, otherwise $\dim_2^{\mathrm{E}}(G) = n - k$;*
2. $\dim_2^{\mathrm{S}}(G) = n - k$;
3. $\dim_2^{\mathrm{J}}(G) = n - k$.

Note that Statement 1 in the Corollary first proved by Roy (2010).

THANK YOU