Representation of graphs: open problems

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Range of the circumradius $\mathcal{R}(G)$

Let $\mathcal{R}(G) < \infty$. *What is the range of* $\mathcal{R}(G)$*?* Since for a fixed n there are finitely many graphs G this range is a countable subset of the interval $[1/\sqrt{2}, \infty)$.

What is the maximum value of $\mathcal{R}(G)$*?*
Can $\mathcal{R}(G)$ *be greater than 1?*

The second distance $\beta_*(G)$

(1) What is the range of $\beta_*(G)$?
(2) Can $\beta_*(G_1) = \beta_*(G_2)$ for distinct G_1 and G_2?

For the second question the answer is positive. Let σ be a collection of positive integers $n_1, \ldots, n_m$ with $m > 1$. We denote

$$|\sigma| := n_1 + \ldots + n_m.$$

Let $\bar{K}_\sigma := \bar{K}_{n_1,\ldots,n_m}$, where $\bar{K}_{n_1,\ldots,n_m}$ is the graph complement of the complete m–partite graph $K_{n_1,\ldots,n_m}$. In other words, $\bar{K}_\sigma$ is the disjoint union of cliques of sizes $n_1, \ldots, n_m$.
Einhorn and Schoenberg proved that $\dim_2^{\mathrm{E}}(\bar{K}_\sigma) = |\sigma| - 1$. The converse statement is also true. If for a graph G on n vertices we have $\dim_2^{\mathrm{E}}(G) = n - 1$, then G is $\bar{K}_\sigma$ for some σ with $|\sigma| = n$.

The second distance $\beta_*(G)$

Let $\sigma_1 = (1, 1, 1)$, $\sigma_2 = (2, 2)$ and $\sigma_3 = (1, 4)$. Then $\beta_*(\sigma_i) = \sqrt{3}$ for $i = 1, 2, 3$.
Another example,

$$\sigma = (1, 1, 1, 1, 1),\ (2, 2, 2),\ (4, 4),\ (2, 8),\ (1, 16).$$

For all these collections $\beta_*(\sigma) = \sqrt{5/2}$.
It is an interesting problem *to describe sets of collections σ with the same $\beta_*(\sigma)$.*

Representations of colored $E(K_n)$ as s–distance sets

First consider an equivalent definition of graph representations. Let $G = (V(G), E(G))$ be a graph on n vertices. We have $E(K_n) = E(G) \cup E(\bar{G})$. Then it is can be considered as a coloring of $E(K_n)$ in two colors. Hence

$$E(K_n) = E_1 \cup E_2, \text{ where } E_1 \cap E_2 = \emptyset.$$

Clearly, G is uniquely defined by the equation $E(G) = E_1$.
Let $L(e) := i$ if $e \in E_i$. Then $L : E(K_n) \to \{1, 2\}$ is a coloring of $E(K_n)$. A representation L as a two–distance set is an embedding f of $V(K_n)$ into $\mathbb{R}^d$ such that $\mathrm{dist}(f(u), f(v)) = a_i$ for $[uv] \in E_i$.
Here $a_2 \geq a_1 > 0$.

Representations of colored $E(K_n)$ as s–distance sets

This definition can be extended to any number of colors. Let $L : E(K_n) \to \{1, \dots, s\}$ be a coloring of the set of edges of a complete graph K_n. Then

$$E(K_n) = E_1 \cup \dots \cup E_s,\ E_i := \{e \in E(K_n) : L(e) = i\}.$$

We say that an embedding f of the vertex set of K_n into $\mathbb{R}^d$ is a *Euclidean representation of a coloring L in $\mathbb{R}^d$ as an s–distance set* if there are s positive real numbers $a_1 \leq \dots \leq a_s$ such that $\mathrm{dist}(f(u), f(v))) = a_i$ if and only if $[uv] \in E_i$.

Representations of colored $E(K_n)$ as s–distance sets

It is easy to extend the definitions of polynomials $C_G(t)$ and $M_G(t)$ for s–distance sets. In this case we have multivariate polynomials $C_L(t_2, \ldots, t_s)$ and $M_L(t_2, \ldots, t_s)$, where $a_1 = 1$ and $t_i = a_i^2$ for $i = 2, \ldots, s$. It is clear that a Euclidean representation of L is spherical only if $F_L(t_2, \ldots, t_s)$ is well defined, where

$$F_L(t_2, \ldots, t_s) := -\frac{1}{2}\frac{M_L(t_2, \ldots, t_s)}{C_L(t_2, \ldots, t_s)}.$$

I think that the Einhorn–Schoenberg theorem and several later results can be generalized for representations of colorings L as s–distance sets.

Contact graph representations of G

The famous circle packing theorem (also known as the Koebe–Andreev–Thurston theorem) states that for every connected simple planar graph G there is a circle packing in the plane whose contact graph is isomorphic to G.

Now consider representations of a graph G as the contact graph of a packing of congruent spheres in $\mathbb{R}^d$.
Equivalently, let X be a finite subset of $\mathbb{R}^d$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

The *contact graph* $\mathrm{CG}(X)$ is a graph with vertices in X and edges (x,y), $x,y \in X$, such that $\mathrm{dist}(x,y) = \psi(X)$. In other words, $\mathrm{CG}(X)$ is the contact graph of a packing of spheres of diameter $\psi(X)$ with centers in X.

Contact graph representations of G

There are several combinatorial properties of contact graphs. For instance, the degree of any vertex of $\mathrm{CG}(X)$, $X \subset \mathbb{R}^d$, is not to exceed the kissing number k_d. For spherical contact graph representations in $\mathbb{S}^2$ this degree is not greater than five.

Using this and other properties of $\mathrm{CG}(X)$ were enumerated spherical irreducible contact graphs for $n \leq 11$ (Musin & Tarasov, 2013). .

It is an interesting problem to *find minimal dimensions of Euclidean and spherical contact graph representations of graphs G.*