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Archiv der Mathematik
Archives Mathématiques Archives of Mathematics

ISSN 0003-889X
Volume 111
Number 5

Arch. Math. (2018) 111:493-501
DOI 10.1007/s00013-018-1237-2

Arch. Math. 111 (2018), 493–501
© 2018 Springer Nature Switzerland AG
0003-889X/18/050493-9
published online August 28, 2018
https://doi.org/10.1007/s00013-018-1237-2

Archiv der Mathematik

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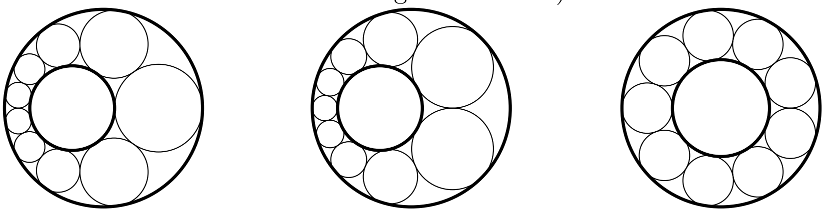
Abstract. In this paper we consider generalizations to higher dimensions of classical results on chains of tangent spheres.

Mathematics Subject Classification. 52C17, 52C26.

Keywords. Steiner's porism, Soddy's Hexlet, Spherical codes.

1. Introduction. Suppose we have a chain of k circles all of which are tangent to two given non-intersecting circles S_1, S_2, and each circle in the chain is tangent to the previous and next circles in the chain. Then, any other circle C that is tangent to S_1 and S_2 along the same bisector is also part of a similar chain of k circles. This fact is known as *Steiner's porism* [1, Chap. 7], [10, Chap. 4, 5]. The usual proof of this is simply to choose an inversion that makes S_1 and S_2 concentric, after which the result follows immediately by rotation symmetry. (Below are shown two closed Steiner chains and the inversion transform to a chain of congruent circles.)

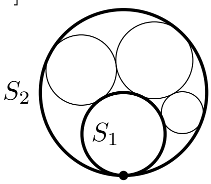


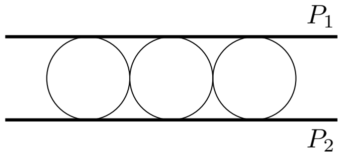
Soddy's hexlet is a chain of six spheres each of which is tangent to both of its neighbors and also to three mutually tangent given spheres. Frederick Soddy published the following theorem in 1937 [11]: "*It is always possible to find a hexlet for any choice of three mutually tangent spheres.*" Note that

The author is partially supported by the NSF Grant DMS-1400876 and the RFBR Grant 15-01-99563.

Soddy's hexlet was also discovered independently in Japan, as shown by Sangaku tablets from 1822 in the Kanagawa prefecture [9].

The general problem of finding a hexlet for three given mutually tangent spheres S_1, S_2, and S_3 can be reduced to the annular case using inversion. Inversion in the point of tangency between spheres S_1 and S_2 transforms them into parallel planes P_1 and P_2. Since sphere S_3 is tangent to both S_1 and S_2 and does not pass through the center of inversion, S_3 is transformed into another sphere S_3' that is tangent to both planes. Six spheres may be packed around S_3' and touch planes P_1 and P_2. Re-inversion restores the three original spheres, and transforms these six spheres into a hexlet for the original problem [1, 10].





Let $\mathcal{F} := \{S_1, S_2\}$, where S_1 and S_2 are tangent spheres in $\mathbb{R}^n$. Let $\Pi_n(\mathcal{F})$ denote the set of all (non-congruent) sphere packings in $\mathbb{R}^n$ such that all spheres in a packing $P \in \Pi_n(\mathcal{F})$ are tangent to both spheres from $\mathcal{F}$. In [7] the authors report that there is an unpublished result by Kirkpatrick and Rote about this case. In fact, they proved that

There is a one-to-one correspondence $T_{\mathcal{F}}$ between sphere packings from $\Pi_n(\mathcal{F})$ and unit sphere packings in $\mathbb{R}^{n-1}$.

It is easy to prove. Indeed, let $T_{\mathcal{F}}$ be an inversion in the point of tangency between spheres from $\mathcal{F}$ such that it makes S_1 and S_2 parallel hyperplanes with the distance between them equals 2. Then the result follows immediately by the fact that a packing $P \in \Pi_n(\mathcal{F})$ transforms to a unit sphere packing $T_{\mathcal{F}}(P)$. ([7, Proposition 4.5] contains a sketch of proof.)

Let X be a set of points in a unit sphere $\mathbb{S}^{d-1}$. We say that X is a *spherical ψ-code* if the angular distance between distinct points in X is at least ψ. Denote by $A(d, \psi)$ the maximal size of a ψ-code in $\mathbb{S}^{d-1}$ [5].

Note that $A(d, \pi/3) = k(d)$, where by $k(d)$ we denote the *kissing number*, i.e. the maximum number of non-overlapping unit spheres in $\mathbb{R}^d$ that can be arranged so that all of them touch one (central) unit sphere.

In this paper we show a relation between sphere packings in $\mathbb{R}^n$ that are tangent spheres in a given family $\mathcal{F}$ and spherical codes (Theorem 2.3). This relation gives generalizations of Steiner's porism and Soddy's hexlet to higher dimensions.

2. $\mathcal{F}$-Kissing arrangements and spherical codes. Here we say that two distinct spheres S_1 and S_2 in $\mathbb{R}^n$ are *non-intersecting* if the intersection of these spheres is not a sphere of radius $r > 0$. In other words, either $S_1 \cap S_2 = \emptyset$ or these spheres touch each other.

Definition 2.1. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be a family of m arbitrary spheres in $\mathbb{R}^n$. (Actually, S_i can be a sphere of any radius or a hyperplane.) We say that a set $\mathcal{C}$ of spheres in $\mathbb{R}^n$ is an $\mathcal{F}$-kissing arrangement if*

(1) *each sphere from $\mathcal{C}$ is tangent to all spheres from $\mathcal{F}$;*
(2) *each sphere from $\mathcal{C}$ is tangent to at least one sphere from $\mathcal{C}$;*
(3) *any two distinct spheres from $\mathcal{C}$ are non-intersecting.*

It is clear that if $\mathcal{C}$ is nonempty and one of spheres from $\mathcal{F}$ contains another, then all S_i as well as all spheres from $\mathcal{C}$ lie in this sphere. If there are no such sphere in $\mathcal{F}$, then depending on radii and arrangements of S_i either one of spheres from $\mathcal{C}$ contains all other from $\mathcal{C}$ and $\mathcal{F}$ or all spheres in $\mathcal{C}$ are non-overlapping.

Definition 2.2. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$, $m \geq 2$, be a family of m spheres in $\mathbb{R}^n$. We say that $\mathcal{F}$ is an S-family if*

(1) *S_1 and S_2 are non-intersecting spheres;*
(2) *each S_i with $i > 2$ can intersect at most one S_j with $j = 1, 2$;*
(3) *there are non-empty $\mathcal{F}$-kissing arrangements and all of them are finite.*

Remark. I wish to thank the anonymous referee of this paper who pointed out that if Definition 2.2 has only assumptions (1) and (3), then $\mathcal{F}$-kissing arrangements are possible can have spheres that touch some spheres in $\mathcal{F}$ from the outside and some from the inside.

Consider the following example. Let $\mathcal{F} := \{S_1, S_2, S_3\}$, where S_1 and S_2 be two concentric spheres (or two parallel hyperplanes) in $\mathbb{R}^n$. Let S_3 be a sphere that intersects S_1 and S_2. Then for some cases there are $\mathcal{F}$-kissing spheres such that some of them are tangent to S_3 from the outside and some from the inside.

However, if we have (2), then there is at most one sphere that is tangent to S_1, S_2, and S_3 from the inside. Indeed, suppose S_3 intersects S_1. Then Definition 2.2(2) yields that S_3 either has no common points with S_2 or S_3 is tangent to S_2 at some point p. In the first case there are no $\mathcal{F}$-kissing spheres that are tangent to S_3 from the inside. It is easy to see that in the second case we can have at most one sphere that is tangent to S_2 and S_3 at p. By Definition 2.1(2) this sphere cannot be a sphere in the $\mathcal{F}$-kissing arrangement.

Note that in Steiner's chain problem, $\mathcal{F}$ consists of two non-intersecting circles S_1 and S_2, and in the problem of finding a hexlet, $\mathcal{F}$ consists of three mutually tangent spheres S_1, S_2, and S_3. Now we consider a general case.

Theorem 2.3. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$, $2 \leq m < n + 2$, be an S-family of m spheres in $\mathbb{R}^n$. Then there is a one-to-one correspondence $\Phi_{\mathcal{F}}$ between $\mathcal{F}$-kissing arrangements and spherical $\psi_{\mathcal{F}}$-codes in $\mathbb{S}^{d-1}$, where $d := n + 2 - m$ and the value $\psi_{\mathcal{F}}$ is uniquely defined by the family $\mathcal{F}$.*

Proof. There are two cases: (i) S_1 and S_2 are tangent or (ii) S_1 and S_2 do not touch each other. In the first case let O be the contact point of these spheres and if we apply the sphere inversion T with center O and an arbitrary radius ρ, then S_1 and S_2 become two parallel hyperplanes S_1' and S_2'. In case (ii) we can use the famous theorem: *It is always possible to invert S_1 and S_2 into a pair of concentric spheres S'_1 and S'_2* (see [10, Theorem 13]).

Let P be an $\mathcal{F}$-kissing arrangement. Since all spheres from P touch S_1 and S_2 after the inversion, they become spheres that touch S'_1 and S'_2. In both cases that yields that all spheres from $P' := T(P)$ are congruent. Without loss of generality, we can assume that spheres from P' are unit. Thus we have a unit sphere packing $P' = \{C'_j\}$ in $\mathbb{R}^n$ such that each sphere C'_j from P' is tangent to all $S'_i := T(S_i)$, $i = 1, \dots, m$.

In case (i) denote by Z_0 the hyperplane of symmetry of S'_1 and S'_2 and in case (ii) Z_0 be a sphere of radius $(r_1 + r_2)/2$ that is concentric with S'_1 and S'_2, where r_i is the radius of S'_i. If $m > 2$, let Z_i, $i = 3, \dots, m$, denote a sphere of radius $(r_i + 1)$ that is concentric with S'_i. Let $S_{\mathcal{F}}$ be the locus of centers of spheres that are tangent to all S'_i. If $m = 2$, the $S_{\mathcal{F}} = Z_0$ and for $m > 2$, $S_{\mathcal{F}}$ is the intersection of spheres Z_0 and Z_i, $i = 3, \dots, m$.

Note that by assumption $S_{\mathcal{F}}$ is not empty. Moreover, since all $\mathcal{F}$-kissing arrangements are finite, $S_{\mathcal{F}}$ is a sphere of radius $r > 0$.

Since all C_j are unit sphere, the distance between centers of distinct spheres in P' is at least 2. Therefore, if $r < 1$, then P contains just one sphere. In this case put for $\psi_{\mathcal{F}}$ any number greater than π.

Now consider the case when $S_{\mathcal{F}}$ is a $(d-1)$-sphere of radius $r \geq 1$. Let $\psi_{\mathcal{F}}$ be the angular distance between centers in $S_{\mathcal{F}}$ of two tangent unit spheres in $\mathbb{R}^n$. In other words, $\psi_{\mathcal{F}}$ is the angle between equal sides in an isosceles triangle with side lengths r, r, and 2. We have

$$\psi_{\mathcal{F}} := \arccos\left(1 - \frac{2}{r^2}\right).$$

Let $f : S_{\mathcal{F}} \to U_{\mathcal{F}}$ be the central projection, where $U_{\mathcal{F}}$ denotes a unit sphere that is concentric with $S_{\mathcal{F}}$. Denote c_P the set of centers of C_j. Let $X := f(c_P)$. Then X is a spherical $\psi_{\mathcal{F}}$-code in $\mathbb{S}^{d-1}$.

Let X be any spherical $\psi_{\mathcal{F}}$-code in $\mathbb{S}^{d-1} \simeq U_{\mathcal{F}}$. Then we have a unit sphere packing Q_X with centers in $c_X := f^{-1}(X)$ such that each sphere from Q_X is tangent to all S'_i. It is clear that $P := T(Q_X)$ is an $\mathcal{F}$-kissing arrangement.

Thus, a one-to-one correspondence $\Phi_{\mathcal{F}}$ between $\mathcal{F}$-kissing arrangements and spherical $\psi_{\mathcal{F}}$-codes in $\mathbb{S}^{d-1}$ is well defined. This completes the proof. □

Corollary 2.4. *Let $\mathcal{F} = \{S_1, \dots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$. Denote by $\mathrm{card}_{\mathcal{F}}$ the maximum cardinality of $\mathcal{F}$-kissing arrangements. Then*

$$\mathrm{card}_{\mathcal{F}} = A(d, \psi_{\mathcal{F}}).$$

In particular, $\mathrm{card}_{\mathcal{F}} \geq d + 1$ if and only if $\psi_{\mathcal{F}} \leq \arccos(-1/d)$.

Proof. The equality $\mathrm{card}_{\mathcal{F}} = A(d, \psi_{\mathcal{F}})$ immediately follows from Theorem 2.3. Since $a_d := \arccos(-1/d)$ is the side length of a regular spherical d-simplex in $\mathbb{S}^{d-1}$, we have $A(d, a_d) = d + 1$. Thus, if $\psi \leq a_d$, then $A(d, \psi) \geq d + 1$. □

Theorem 2.3 states a one-to-one correspondence between $\mathcal{F}$-kissing arrangements and spherical codes. We say that *two $\mathcal{F}$-kissing arrangements M and N*

are equivalent if the correspondent spherical $\psi_{\mathcal{F}}$-codes X and Y are isometric in $\mathbb{S}^{d-1}$.

Theorem 2.5. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$. Then any $\psi_{\mathcal{F}}$-code X in $\mathbb{S}^{d-1}$ uniquely determines the set of equivalent $\mathcal{F}$-kissing arrangements $\{P_A(X)\}$ such that this set can be parametrized by $A \in \mathrm{SO}(d)$. Moreover, for any isometric $\psi_{\mathcal{F}}$-codes X and Y in $\mathbb{S}^{d-1}$ and $A, B \in \mathrm{SO}(d)$, $P_A(X)$ can be transform to $P_B(Y)$ by a conformal map.*

Proof. Here we use the same notations as in the proof of Theorem 2.3.

Denote $P = T(Q_X)$ by P_I, where I is the identity element in $\mathrm{SO}(d)$. If $\psi_{\mathcal{F}}$-codes X and Y are isometric in $\mathbb{S}^{d-1}$, then there is $A \in \mathrm{SO}(d)$ such that $Y = A(X)$. Denote $T(Q_A)$ by P_A. We have

$$P_A = h_A(P_I), \quad h_A := T \circ A \circ T.$$

It is clear that h_A is a conformal map. □

3. Analogs of Steiner's porism and Soddy's hexlet.

3.1. Analogs of Steiner's porism. Theorem 2.5 can be considered as a generalization of Steiner's porism. For a given family $\mathcal{F}$ and spherical $\psi_{\mathcal{F}}$-code X in $\mathbb{S}^{d-1}$, there are $\mathcal{F}$-kissing arrangements that are correspondent to X.

However, Steiner's porism has a stronger property. A Steiner chain is formed from one starting circle and *each circle in the chain is tangent to the previous and next circles in the chain.* If the last circle touches the first, this will also happen for any position of the first circle. Thus, a position of the first circle uniquely determines a Steiner chain.

Now we extend this property to higher dimensions. We say that an $\mathcal{F}$-kissing arrangement $\mathcal{C} = \{C_1, \ldots, C_k\}$ is a *k-clique* if all spheres in $\mathcal{C}$ are mutually tangent. We say that a sphere C_{k+1} is *adjacent* to $\mathcal{C}$ if C_{k+1} is tangent to all spheres of $\mathcal{C}$ and $\mathcal{F}$.

Lemma 3.1. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$ with $\mathrm{card}_{\mathcal{F}} \geq d+1$. Then the set of $(d-1)$-cliques is not empty and for any $(d-1)$-clique $\mathcal{C}$ there are exactly two adjacent spheres.*

Proof. Corollary 2.4 yields that $\psi_{\mathcal{F}} \leq \arccos(-1/d)$. Therefore, a regular spherical $(d-2)$-simplex of side length $\psi_{\mathcal{F}}$ can be embedded into $\mathbb{S}^{d-1}$. For this simplex in $\mathbb{S}^{d-1}$ there are exactly two possibilities to complete it to regular spherical $(d-1)$-simplices. By Theorem 2.3 these two new vertices correspond to two adjacent spheres. □

Now we define a Steiner arrangement for all dimensions. First we define a tight $\mathcal{F}$-kissing arrangement, where $\mathcal{F}$ is an S-family of spheres in $\mathbb{R}^n$. Let $\mathcal{C}_0$ be any $(d-1)$-clique. By Lemma 3.1 there are two adjacent spheres for $\mathcal{C}_0$. Let C_1 be one of them. Then $\mathcal{C}_1 := \mathcal{C}_0 \cup C_1$ is a d-clique of tangent spheres. Suppose that after k steps we have an $\mathcal{F}$-kissing arrangement $\mathcal{C}_k$. We can do the next step only if in $\mathcal{C}_k$ there are a $(d-1)$-clique and its adjacent sphere C_{k+1} such that $\mathcal{C}_{k+1} := \mathcal{C}_k \cup C_{k+1}$ is an $\mathcal{F}$-kissing arrangement. Denote by t

the maximum number of possible steps. It is clear, $t \leq \mathrm{card}_{\mathcal{F}}$. We call $\mathcal{C}_t$ a *tight $\mathcal{F}$-kissing arrangement*.

Note that for $d = 2$ a tight chain $\mathcal{C}_t$ is Steiner if the first circle of the chain touches the last one. It can be extended for all dimensions. We say that a tight $\mathcal{F}$-kissing arrangement $\mathcal{C}_t$ is *Steiner* if $\mathcal{C}_t$ contains all adjacent spheres of all its $(d-1)$-cliques. Equivalently, an $\mathcal{F}$-Steiner arrangement can be define in the following way.

Definition 3.2. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$ with* $\mathrm{card}_{\mathcal{F}} \geq d+1$. *We say that an $\mathcal{F}$-kissing arrangement $\mathcal{C}$ is Steiner if it contains a $(d-1)$-clique and for all $(d-1)$-cliques in $\mathcal{C}$ their adjacent spheres also lie in $\mathcal{C}$.*

Recall that a simplicial polytope is a polytope whose facets are all simplices.

Definition 3.3. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$. An $\mathcal{F}$-kissing arrangement is called $(d-1)$-simplicial if the convex hull of the correspondent spherical code in $\mathbb{S}^{d-1}$ is a $(d-1)$-simplicial regular polytope. We denote this polytope by $P_{\mathcal{F}}$.*

Lemma 3.4. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$ with* $\mathrm{card}_{\mathcal{F}} \geq d+1$. *An $\mathcal{F}$-kissing arrangement is Steiner if and only if it is $(d-1)$-simplicial.*

Proof. Clearly, if an $\mathcal{F}$-kissing arrangement $\mathcal{C}$ is simplicial, then it is Steiner. Suppose $\mathcal{C}$ is Steiner. Then the convex hull P of the correspondent spherical $\psi_{\mathcal{F}}$-code $\Phi_{\mathcal{F}}(\mathcal{C})$ (see Theorem 2.3) is a polytope that has a $(d-2)$-face P_0 which is a regular $(d-2)$-simplex of side length $\psi_{\mathcal{F}}$. By Lemma 3.1, P_0 has two adjacent points v_0 and v_1 in $\mathbb{S}^{d-1}$. Moreover, by Definition 3.2, these two points are vertices of P. Therefore all vertices of a bipyramid $P_1 := v_0 \cup P_0 \cup v_1$ are vertices of P. It is clear that all faces of P_1 are regular $(d-2)$-simplices, i.e. they are $(d-2)$-cliques in $\mathcal{C}$. It yields that P_1 a sub-polytope of P. Next, we add all new adjacent vertices to $(d-2)$-faces of P_1. We denote this sub-polytope of P by P_2. We can continue this process and define new P_i. It is easy to see that after finitely many steps we obtain $P_k = P$.

Note that for $i > 0$ any P_i consists of regular $(d-1)$-simplices of side length $\psi_{\mathcal{F}}$. Then all faces of P are regular simplices. Since P is a spherical polytope, we have that P is regular. □

Theorem 3.5. *Let $\mathcal{F} = \{S_1, \ldots, S_m\}$ be an S-family of spheres in $\mathbb{R}^n$. If for $\mathcal{F}$ there exists a Steiner arrangement, then we have one of the following cases*

1. *$d = 2$, $\psi_{\mathcal{F}} = 2\pi/k$, $k \geq 3$, and $P_{\mathcal{F}}$ is a regular polygon with k vertices.*
2. *$\psi_{\mathcal{F}} = \arccos(-1/d)$ and $P_{\mathcal{F}}$ is a regular d-simplex with any $d \geq 2$.*
3. *$\psi_{\mathcal{F}} = \pi/2$ and $P_{\mathcal{F}}$ is a regular d-crosspolytope with any $d \geq 2$.*
4. *$d = 3$, $\psi_{\mathcal{F}} = \arccos(1/\sqrt{5})$, and $P_{\mathcal{F}}$ is a regular icosahedron.*
5. *$d = 4$, $\psi_{\mathcal{F}} = \pi/5$, and $P_{\mathcal{F}}$ is a regular 600-cell.*

Proof. Lemma 3.4 reduces a classification of Steiner arrangements to an enumeration of simplicial regular polytopes. The list of these polytopes is well known, see [6], and it is as in the theorem. □

In particular, Theorem 3.5 shows that for any of these five cases all tight $\mathcal{F}$-kissing arrangements are equivalent. The following corollary is a generalization of Steiner's porism.

Corollary 3.6. *Let $\mathcal{F}$ be an S-family of spheres in $\mathbb{R}^n$. If there is an $\mathcal{F}$-Steiner arrangement, then any tight $\mathcal{F}$-kissing arrangement is Steiner.*

3.2. Analogs of Soddy's hexlet. Soddy [11] proved that for any family $\mathcal{F}$ of three mutually tangent spheres in $\mathbb{R}^3$ there is a chain of six spheres (hexlet) such that each sphere from this chain is tangent all spheres from $\mathcal{F}$. Now we extend this theorem to higher dimensions.

Let $m \geq 2$. Denote

$$\psi_m := \arccos\left(\frac{1}{m-1}\right).$$

Theorem 3.7. *Let $3 \leq m < n+2$. Let X be a spherical ψ_m-code in $\mathbb{S}^{d-1}$, where $d := n+2-m$. Then for any family $\mathcal{F}$ of m mutually tangent spheres in $\mathbb{R}^n$, there is an $\mathcal{F}$-kissing arrangement that corresponds to X.*

Proof. Here we use the same notations as in the proof of Theorem 2.3.

This theorem follows from Theorem 2.3 using the fact that $\psi_{\mathcal{F}} = \psi_m$. Indeed, we have case (i), and therefore S_1' and S_2' are two parallel hyperplanes. Since the spheres in $\mathcal{F}$ are mutually tangent, we have that all S_i', $i = 3, \ldots, m$, are unit spheres. It is not hard to prove that $S_{\mathcal{F}}$ is the intersection of $(m-2)$ spheres of radius 2 centered at points $C = \{c_3, \ldots, c_m\}$ in $\mathbb{R}^{n-1}$ such that if $m > 3$, then $\operatorname{dist}(c_i, c_j) = 2$ for all distinct c_i and c_j from C. Then

$$r = \sqrt{\frac{2m-2}{m-2}} \quad \text{and} \quad 1 - \frac{2}{r^2} = \frac{1}{m-1}.$$

It proves the equality $\psi_{\mathcal{F}} = \psi_m$. □

Let $\mathcal{F}$ be a family of m mutually tangent spheres in $\mathbb{R}^n$. Denote

$$S(n,m) := \operatorname{card}_{\mathcal{F}}.$$

Corollary 2.4 and Theorem 3.7 imply

Corollary 3.8. $S(n,m) = A(n+2-m, \psi_m)$. *In particular,* $S(n,3) = k(n-1)$.

Examples.

1. If $m = n+1$, then $\psi_m = \pi$. It implies that $S(n, n+1) = 2$. Actually, this fact can be proved directly, there are just two spheres that are tangent to $n+1$ mutually tangent spheres in $\mathbb{R}^n$.
2. Now consider a classical case $m = n = 3$. We have

$$S(3,3) = A(2, \pi/3) = k(2) = 6.$$

Then a maximum $\pi/3$-code in $\mathbb{S}^1$ is a regular *hexagon*. The corresponding $\mathcal{F}$-sphere arrangement is a Soddy's hexlet.

3. Let $m = 3$ and X be a spherical code of maximum cardinality $|X| = k(d)$, where $d := n-1$. Then X is a *kissing arrangement* (maximum $\pi/3$-code) in $\mathbb{S}^{d-1}$. Note that the kissing number problem has been solved only for $n \leq 4$, $n = 8$, and $n = 24$ (see [2,5,8]). However, in several dimensions many nice kissing arrangements are known, for instance, in dimensions 8 and 24 [5].
   If $n = 4$, i.e. $d = 3$, then $k(d) = 12$. In this dimension there are infinitely many non-isometric kissing arrangements. We think that the *cuboctahedron* with 12 vertices representing the positions of 12 neighboring spheres can be a good analog of Soddy's hexlet in four dimensions.
   In four dimensions the kissing number is 24 and the best known kissing arrangement is a regular 24-*cell* [8]. (However, the conjecture about uniqueness of this kissing arrangement is still open.) So in dimension five a nice analog of Soddy's hexlet is the 24-cell.
4. By Theorem 3.7 for $\mathcal{F}$-kissing arrangements correspondent spherical codes have to have the inner product $= 1/(m-1)$. The book [5] contains a large list of such spherical codes. Moreover, some of them are universally optimal [4, Table 1]. All these examples give analogs of Soddy's hexlet in higher dimensions.
5. Hao Chen [3, Sect. 3] considers sphere packings for some graph joins. [3, Table 1] contains a large list of spherical codes that give generalizations of Soddy's hexlet.

Acknowledgements. I wish to thank Arseniy Akopyan and Alexey Glazyrin for useful comments and references.

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Received: 30 January 2018