On the number of discrete chains

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LSE and MIPT

Joint work with Andrey Kupavskii

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Unit distances

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$n^{1+c/\log\log n} \le u_2(n) \le Cn^{4/3}$ [Erdős'46, Spencer-Szemerédi-Trotter'84]

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Note that $u_d(n) = \Theta(n^2)$ if $d \geq 4$.

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From now on $d = 2$.

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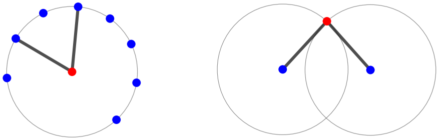

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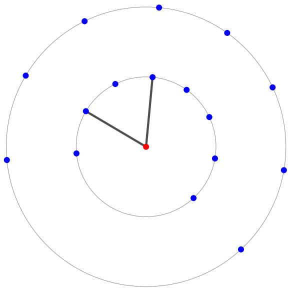

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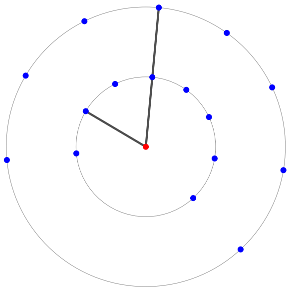

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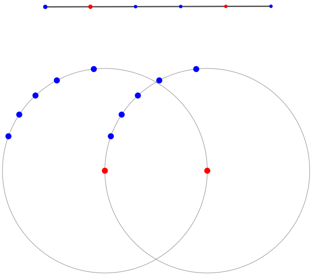

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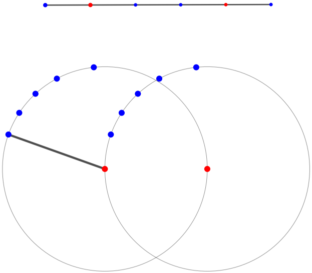

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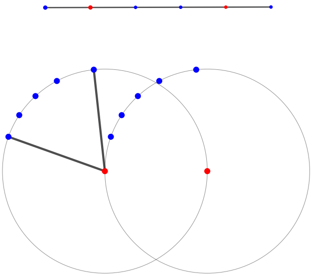

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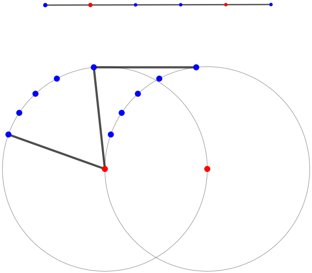

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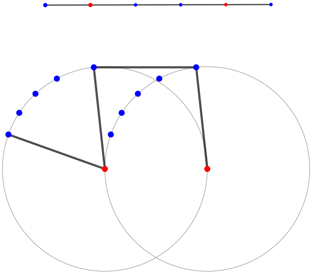

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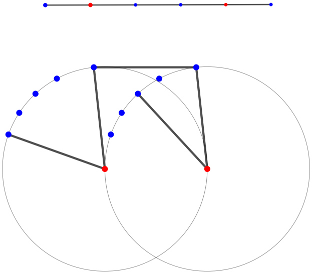

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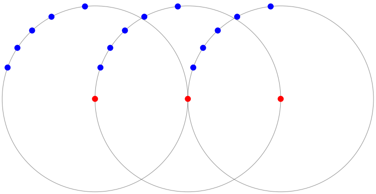

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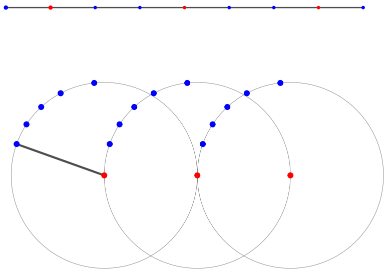

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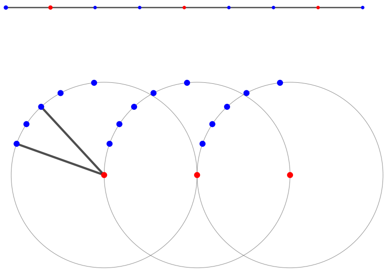

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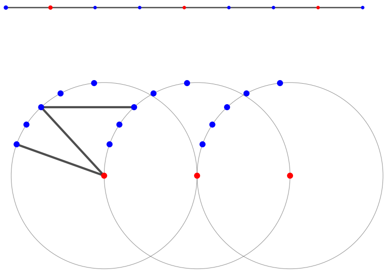

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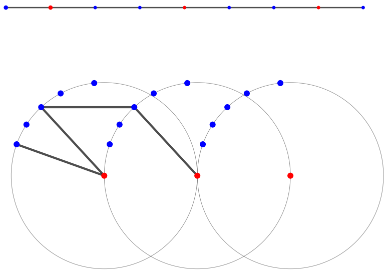

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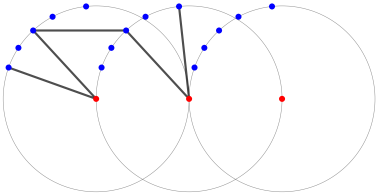

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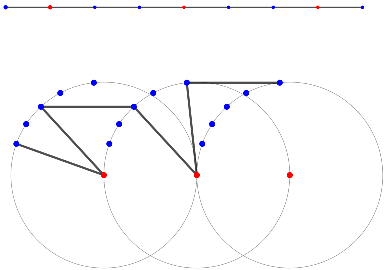

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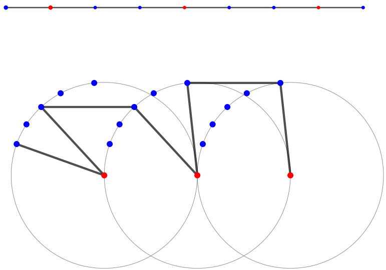

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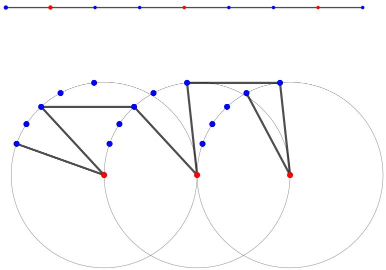

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$$U_k(n) = \begin{cases} O\left(n \cdot u(n)^{k/3}\right) & \text{if } k \equiv 0 \pmod 3 \\ O\left(u(n)^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3 \\ O\left(n^2 \cdot u(n)^{(k-2)/3}\right) & \text{if } k \equiv 2 \pmod 3 \end{cases}$$ [PSS'19]

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$U_k(n) = O\left(n^{2k/5 + 1 + \gamma(k)}\right)$ [PSS'19]

$(k, \boldsymbol{\delta})$-chains

For $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ a $(k+1)$-tuple $(p_1, \ldots, p_{k+1})$ of distinct points is a $(k, \boldsymbol{\delta})$-chain if $\|p_i - p_{i+1}\| = \delta_i$ for all $i = 1, \ldots, k$.

$C_k(n)$ = the maximum number of $(k, \boldsymbol{\delta})$-chains in a set of n points, where the maximum is taken over all $\boldsymbol{\delta}$.

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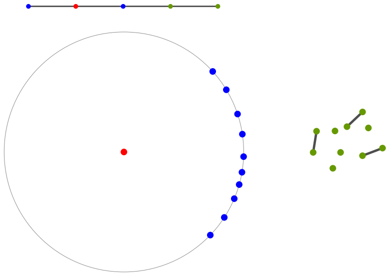

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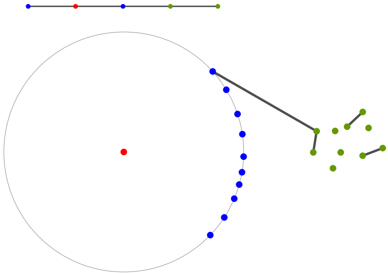

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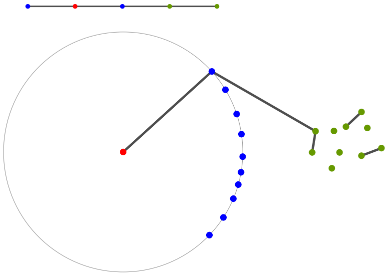

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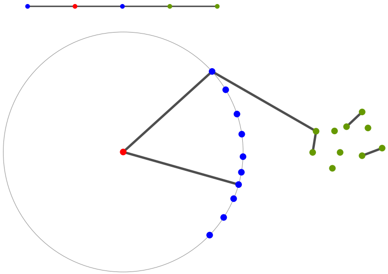

Bounds on the number of chains

Clearly $C_k(n) \geq U_k(n)$.

By the constructions from the previous slides:
$C_k(n) = \Omega\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right)$
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Theorem (F.-Kupavskii)

$$C_k(n) = \begin{cases} \tilde{O}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) & \text{if } k \equiv 0, 2 \pmod 3 \\ O\left(n^{(k-1)/3+\varepsilon} u(n)\right) & \text{if } k \equiv 1 \pmod 3 \end{cases}$$

Multipartite version

For simplicity we will only bound $U_k(n)$.

$U_k(P_1, \ldots, P_{k+1}) = |\{(p_1, \ldots, p_{k+1}) : \|p_i - p_{i+1}\| = 1 \text{ and } p_i \in P_i\}|$

$U_k(n_1, \ldots, n_{k+1}) = \max_{|P_i| = n_i} U_k(P_1, \ldots, P_{k+1})$

It is enough to bound $U_k(n, \ldots, n)$, since we have

$$U_k(n) \leq U_k(n, \ldots, n) \leq U_k\left((k+1)n\right).$$

The $k \equiv 2 \pmod 3$ case

Claim

For every k and $x \in [0,1]$ we have

$$U_k(n, n, \dots, n, n^x) = \tilde{O}\left(n^{\frac{k+x}{3}+1}\right).$$

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$$U_k(n) = \begin{cases} \tilde{O}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) & \text{if } k \equiv 0, 2 \pmod 3 \\ O\left(n^{(k-1)/3+\varepsilon} u(n)\right) & \text{if } k \equiv 1 \pmod 3 \end{cases}$$

Preparation: Rich points

A point p is n^{α}-rich with respect to a set P if there are at least n^{α} points in P at distance 1 from p.

If $|P| = n^{x}$, then the number of points that are n^{α}-rich with respect to P is $O\left(n^{2x-3\alpha} + n^{x-\alpha}\right)$. [Spencer-Szemerédi-Trotter]

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For $\alpha \geq \frac{x}{2}$: $O\left(n^{2x-3\alpha} + n^{x-\alpha}\right) = O\left(n^{x-\alpha}\right)$

For $\alpha \leq \frac{x}{2}$: $O\left(n^{2x-3\alpha} + n^{x-\alpha}\right) = O\left(n^{2x-3\alpha}\right)$

Proof of the claim

Claim

For every k and $x \in [0,1]$ we have $U_k(n, n, \ldots, n, n^x) = \tilde{O}\left(n^{\frac{k+x}{3}+1}\right)$.

Proof.

Let $P_1, \ldots, P_{k+1}$ be such that $|P_1| = \cdots = |P_k| = n$, $|P_{k+1}| = n^x$ and $U_k(P_1, \ldots, P_k, P_{k+1}) = U_k(n, \ldots, n, n^x)$.

For $\alpha \in [0, x]$ let $P^\alpha \subseteq P_k$ be the set of those points that are at least n^α-rich w.r.t. P_{k+1}, but at most $2n^\alpha$-rich.

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We will show that $U_k(P_1, \ldots, P^\alpha, P_{k+1}) = \tilde{O}\left(n^{\frac{k+x}{3}+1}\right)$ for any α.

This is sufficient, because

$$U_k(P_1, \ldots, P_{k+1}) = \bigcup_{\alpha \in \Lambda} U_k(P_1, \ldots, P^\alpha, P_k) \text{ with } |\Lambda| = \log n.$$

□

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$U_k(P_1, \ldots, P^\alpha, P_{k+1}) = \tilde{O}\left(n^{\frac{k+x}{3}+1}\right)$ *for any* α*.*

Proof.

Case 1: $\alpha \geq \frac{x}{2}$**.**

Then $|P^\alpha| = O(n^{x-\alpha})$, thus $U_1(P^\alpha, P_{k+1}) = O(n^x)$. We obtain

$$U_k(P_1, \ldots, P^\alpha, P_{k+1}) \leq U_{k-3}(P_1, \ldots, P_{k-2}) U_1(P^\alpha, P_{k+1}) = \tilde{O}\left(n^{\frac{k+1}{3}} n^x\right)$$

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Then $|P^\alpha| = O(n^{2x-3\alpha})$. We obtain

$$U_k(P_1, \ldots, P^\alpha, P_{k+1}) \leq U_{k-1}(P_1, \ldots, P^\alpha) 2n^\alpha = \tilde{O}\left(n^{\frac{k-1+2x-3\alpha}{3}+1} n^\alpha\right)$$

□

The $k \equiv 0 \pmod 3$ case

Claim

For every k and $x, y \in [0,1]$ we have

$$U_k(n^x, n, \ldots, n, n^y) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right),$$

where $f(k) = k+2$ if $k \equiv 2 \pmod 3$ and $f(k) = k+1$ otherwise.

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Proof sketch.

We take

$$U_k(P_1, \ldots, P_{k+1}) = \bigcup_{\alpha,\beta\in\Lambda} U_k(P_1, P^\beta, \ldots, P^\alpha, P_k)$$

with $|\Lambda| = \log^2 n$.

We show that $U_k(P_1, P^\beta, \ldots, P^\alpha, P_k) = \tilde{O}\left(n^{\frac{f(k)+x+y}{3}}\right)$ □

The $k \equiv 1 \pmod 3$ case

Theorem (F.-Kupavskii)

$$U_k(n) = \begin{cases} \tilde{O}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) & \text{if } k \equiv 0, 2 \pmod 3 \\ O\left(n^{(k-1)/3+\varepsilon} u(n)\right) & \text{if } k \equiv 1 \pmod 3 \end{cases}$$

For $k \equiv 1 \pmod 3$ it is difficult to work with $U_k(n^x, n, \ldots, n, n^y)$.

An inductive statement would involve $\max\{U_2(n^x, n), U_2(n^y, n)\}$.

The change of $U_2(n^x, n)$ as x is increasing, is not well understood.

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We use a more complicated decomposition

$$U_k(P_1, \ldots, P_k) = \bigcup_{\Lambda} U_k(Q_1, \ldots, Q_k).$$

For $k = 4$ we use $|\Lambda| = \tilde{O}(1)$ parts, such that between consecutive parts we have 'regularity' from left to the right.

For larger $k \equiv 1 \pmod 3$ we use $|\Lambda| = O(n^{\varepsilon})$ parts, such that between consecutive parts we have 'regularity' in both directions.

Paths in $\mathbb{R}^3$

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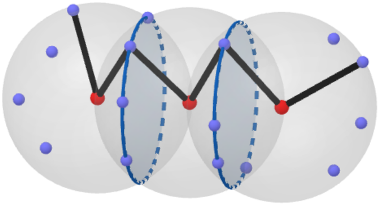

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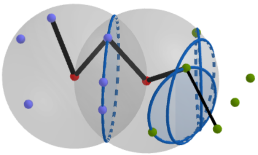
Improved lower bound for odd k:

$$U_k^3(n) = \Omega\left(\max\left\{\frac{u_3(n)^k}{n^{k-1}}, us_3(n)n^{(k-1)/2}\right\}\right).$$

$us_3(n)$= max number of unit distances between a set of n points in $\mathbb{R}^3$ and a set of n points on the sphere = max number of incidences between a set of n points and a set of n circles in the plane



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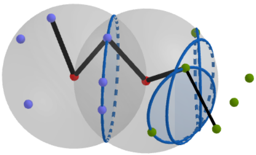

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Thank you for watching.