Parametric approach to Khintchine's transference theorem and its generalizations

Oleg N. German

Moscow State University
Department of Number Theory

Combinatorics and Geometry Days II, Apr 15, 2020

Dirichlet's theorem (1842)

A single number $\theta \in \mathbb{R}$

The inequality $|\theta x - y| \leqslant |x|^{-1}$ admits ∞ solutions in $(x, y) \in \mathbb{Z}^2$, $x \neq 0$

Dirichlet's theorem (1842)

A single number $\theta \in \mathbb{R}$

The inequality $|\theta x - y| \leqslant |x|^{-1}$ admits ∞ solutions in $(x, y) \in \mathbb{Z}^2$, $x \neq 0$

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad |\mathbf{y}| = \max_{1 \leqslant i \leqslant n} |y_i|$$

Dirichlet's theorem (1842)

A single number $\theta \in \mathbb{R}$

The inequality $|\theta x - y| \leqslant |x|^{-1}$ admits ∞ solutions in $(x, y) \in \mathbb{Z}^2$, $x \neq 0$

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad |\mathbf{y}| = \max_{1 \leqslant i \leqslant n} |y_i|$$

Simultaneous approximation to $\theta_1, \ldots, \theta_n$

The inequality $|\Theta x - \mathbf{y}| \leqslant |x|^{-1/n}$ admits ∞ solutions in $(x, \mathbf{y}) \in \mathbb{Z}^{n+1}$, $x \neq 0$

Linear form $x + \theta_1 y_1 + \ldots + \theta_n y_n$

The inequality $|x + \Theta^\intercal \mathbf{y}| \leqslant |\mathbf{y}|^{-n}$ admits ∞ solutions in $(x, \mathbf{y}) \in \mathbb{Z}^{n+1}$, $\mathbf{y} \neq \mathbf{0}$

Khintchine's transference principle (1926)

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad |\mathbf{y}| = \max_{1 \leqslant i \leqslant n} |y_i|$$

Diophantine exponent for Θ

$$\omega(\Theta) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}| \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0 \right\}$$

Diophantine exponent for $\Theta^\intercal$

$$\omega(\Theta^\intercal) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; |x + \Theta^\intercal \mathbf{y}| \leqslant |\mathbf{y}|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ \mathbf{y} \neq \mathbf{0} \right\}$$

Khintchine's transference principle (1926)

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad |\mathbf{y}| = \max_{1 \leqslant i \leqslant n} |y_i|$$

Diophantine exponent for Θ

$$\omega(\Theta) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}| \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0 \right\}$$

Diophantine exponent for $\Theta^\intercal$

$$\omega(\Theta^\intercal) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; |x + \Theta^\intercal \mathbf{y}| \leqslant |\mathbf{y}|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ \mathbf{y} \neq \mathbf{0} \right\}$$

Khintchine's theorem (1926)

$$\omega(\Theta^\intercal) \geqslant n - 1 + n\omega(\Theta)$$

$$\omega(\Theta)^{-1} \leqslant n - 1 + n\omega(\Theta^\intercal)^{-1}$$

Parametric geometry of numbers (2009, by Schmidt and Summerer)

A lattice, a box, and the space of parameters

- Λ a unimodular lattice in $\mathbb{R}^d$
- $\mathcal{B} = \left\{ \mathbf{x} \in \mathbb{R}^d \,\middle|\, |\mathbf{x}| \leqslant 1 \right\}$
- $\mathcal{T} = \left\{ \boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \,\middle|\, \tau_1 + \ldots + \tau_d = 0 \right\}$

Successive minima

- $\mathcal{B}_{\boldsymbol{\tau}} = \operatorname{diag}(e^{\tau_1}, \ldots, e^{\tau_d})\mathcal{B}, \qquad \boldsymbol{\tau} \in \mathcal{T}$
- $\lambda_k(\mathcal{B}_{\boldsymbol{\tau}}) = \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}, \Lambda) = \inf \left\{ \lambda > 0 \,\middle|\, \lambda\mathcal{B}_{\boldsymbol{\tau}} \text{ contains } k \text{ linearly independent points of } \Lambda \right\}$
- $S_k(\boldsymbol{\tau}) = S_k(\boldsymbol{\tau}, \Lambda) = \sum_{1 \leqslant j \leqslant k} \log\left(\lambda_j(\mathcal{B}_{\boldsymbol{\tau}})\right), \qquad k = 1, \ldots, d$

Main question

Behaviour of $S_k(\boldsymbol{\tau})$ as $\boldsymbol{\tau} \to \infty$

Parametric geometry of numbers: local effects

Two classical results

- Minkowski's second theorem (1896): $\frac{1}{d!} \leqslant \prod_{1 \leqslant k \leqslant d} \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}) \leqslant 1$
- Mahler's theorem (1939): $1 \leqslant \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}, \Lambda)\lambda_{d+1-k}(\mathcal{B}_{\boldsymbol{\tau}}^{\circ}, \Lambda^*) \leqslant d!$

Parametric geometry of numbers: local effects

Two classical results

- Minkowski's second theorem (1896): $\frac{1}{d!} \leqslant \prod_{1 \leqslant k \leqslant d} \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}) \leqslant 1$
- Mahler's theorem (1939): $1 \leqslant \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}, \Lambda)\lambda_{d+1-k}(\mathcal{B}_{\boldsymbol{\tau}}^{\circ}, \Lambda^*) \leqslant d!$

Local relations

- $S_1(\boldsymbol{\tau}, \Lambda) \leqslant \ldots \leqslant \frac{S_k(\boldsymbol{\tau}, \Lambda)}{k} \leqslant \ldots \leqslant \frac{S_{d-1}(\boldsymbol{\tau}, \Lambda)}{d-1} = \frac{S_1(-\boldsymbol{\tau}, \Lambda^*)}{d-1} + O(1)$
- $\frac{S_1(\boldsymbol{\tau}, \Lambda)}{d-1} \geqslant \ldots \geqslant \frac{S_k(\boldsymbol{\tau}, \Lambda)}{d-k} \geqslant \ldots \geqslant S_{d-1}(\boldsymbol{\tau}, \Lambda) = S_1(-\boldsymbol{\tau}, \Lambda^*) + O(1)$

Parametric geometry of numbers: local effects

Two classical results

- Minkowski's second theorem (1896): $\dfrac{1}{d!} \leqslant \prod\limits_{1\leqslant k\leqslant d} \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}) \leqslant 1$
- Mahler's theorem (1939): $1 \leqslant \lambda_k(\mathcal{B}_{\boldsymbol{\tau}}, \Lambda)\lambda_{d+1-k}(\mathcal{B}^{\circ}_{\boldsymbol{\tau}}, \Lambda^*) \leqslant d!$

Local relations

- $S_1(\boldsymbol{\tau}, \Lambda) \leqslant \ldots \leqslant \dfrac{S_k(\boldsymbol{\tau}, \Lambda)}{k} \leqslant \ldots \leqslant \dfrac{S_{d-1}(\boldsymbol{\tau}, \Lambda)}{d-1} = \dfrac{S_1(-\boldsymbol{\tau}, \Lambda^*)}{d-1} + O(1)$
- $\dfrac{S_1(\boldsymbol{\tau}, \Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{S_k(\boldsymbol{\tau}, \Lambda)}{d-k} \geqslant \ldots \geqslant S_{d-1}(\boldsymbol{\tau}, \Lambda) = S_1(-\boldsymbol{\tau}, \Lambda^*) + O(1)$

Main question

Behaviour of $S_k(\boldsymbol{\tau})$ as $\boldsymbol{\tau} \to \infty$

Parametric geometry of numbers: Schmidt–Summerer exponents

Dynamics along a path

- space of parameters $\mathcal{T}=\left\{\boldsymbol{\tau}=(\tau_1,\ldots,\tau_d)\in\mathbb{R}^d \,\middle|\, \tau_1+\ldots+\tau_d=0\right\}$
- (quite) arbitrary path $\mathfrak{T}:\ \mathbb{R}_+\to\mathcal{T},\quad \mathfrak{T}:\ s\mapsto\boldsymbol{\tau}(s)$
- Schmidt–Summerer exponents $\Psi_k(\mathfrak{T},\Lambda)=\liminf\limits_{s\to+\infty}\dfrac{S_k(\boldsymbol{\tau}(s))}{s}$

Parametric geometry of numbers: Schmidt–Summerer exponents

Dynamics along a path

- space of parameters $\mathcal{T} = \left\{ \boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \,\middle|\, \tau_1 + \ldots + \tau_d = 0 \right\}$
- (quite) arbitrary path $\mathfrak{T} : \mathbb{R}_+ \to \mathcal{T}$, $\mathfrak{T} : s \mapsto \boldsymbol{\tau}(s)$
- Schmidt–Summerer exponents $\Psi_k(\mathfrak{T}, \Lambda) = \liminf\limits_{s \to +\infty} \dfrac{S_k(\boldsymbol{\tau}(s))}{s}$

Local relations

- $S_1(\boldsymbol{\tau}, \Lambda) \leqslant \ldots \leqslant \dfrac{S_k(\boldsymbol{\tau}, \Lambda)}{k} \leqslant \ldots \leqslant \dfrac{S_{d-1}(\boldsymbol{\tau}, \Lambda)}{d-1} = \dfrac{S_1(-\boldsymbol{\tau}, \Lambda^*)}{d-1} + O(1)$
- $\dfrac{S_1(\boldsymbol{\tau}, \Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{S_k(\boldsymbol{\tau}, \Lambda)}{d-k} \geqslant \ldots \geqslant S_{d-1}(\boldsymbol{\tau}, \Lambda) = S_1(-\boldsymbol{\tau}, \Lambda^*) + O(1)$

Parametric geometry of numbers: Schmidt–Summerer exponents

Dynamics along a path

- space of parameters $\mathcal{T}=\left\{\boldsymbol{\tau}=(\tau_1,\ldots,\tau_d)\in\mathbb{R}^d\,\middle|\,\tau_1+\ldots+\tau_d=0\right\}$
- (quite) arbitrary path $\mathfrak{T}:\ \mathbb{R}_+\to\mathcal{T},\quad \mathfrak{T}:\ s\mapsto\boldsymbol{\tau}(s)$
- Schmidt–Summerer exponents $\Psi_k(\mathfrak{T},\Lambda)=\liminf\limits_{s\to+\infty}\dfrac{S_k(\boldsymbol{\tau}(s))}{s}$

Local relations

- $S_1(\boldsymbol{\tau},\Lambda)\leqslant\ldots\leqslant\dfrac{S_k(\boldsymbol{\tau},\Lambda)}{k}\leqslant\ldots\leqslant\dfrac{S_{d-1}(\boldsymbol{\tau},\Lambda)}{d-1}=\dfrac{S_1(-\boldsymbol{\tau},\Lambda^*)}{d-1}+O(1)$
- $\dfrac{S_1(\boldsymbol{\tau},\Lambda)}{d-1}\geqslant\ldots\geqslant\dfrac{S_k(\boldsymbol{\tau},\Lambda)}{d-k}\geqslant\ldots\geqslant S_{d-1}(\boldsymbol{\tau},\Lambda)=S_1(-\boldsymbol{\tau},\Lambda^*)+O(1)$

Inequalities for the exponents

- $\Psi_1(\mathfrak{T},\Lambda)\leqslant\ldots\leqslant\dfrac{\Psi_k(\mathfrak{T},\Lambda)}{k}\leqslant\ldots\leqslant\dfrac{\Psi_{d-1}(\mathfrak{T},\Lambda)}{d-1}=\dfrac{\Psi_1(-\mathfrak{T},\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T},\Lambda)}{d-1}\geqslant\ldots\geqslant\dfrac{\Psi_k(\mathfrak{T},\Lambda)}{d-k}\geqslant\ldots\geqslant\Psi_{d-1}(\mathfrak{T},\Lambda)=\Psi_1(-\mathfrak{T},\Lambda^*)$

Inequalities for the exponents

- $\Psi_1(\mathfrak{T},\Lambda) \leqslant \ldots \leqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{k} \leqslant \ldots \leqslant \dfrac{\Psi_{d-1}(\mathfrak{T},\Lambda)}{d-1} = \dfrac{\Psi_1(-\mathfrak{T},\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T},\Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{d-k} \geqslant \ldots \geqslant \Psi_{d-1}(\mathfrak{T},\Lambda) = \Psi_1(-\mathfrak{T},\Lambda^*)$

Parametric geometry of numbers: Khintchine's transference theorem

Inequalities for the exponents

- $\Psi_1(\mathfrak{T},\Lambda) \leqslant \ldots \leqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{k} \leqslant \ldots \leqslant \dfrac{\Psi_{d-1}(\mathfrak{T},\Lambda)}{d-1} = \dfrac{\Psi_1(-\mathfrak{T},\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T},\Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{d-k} \geqslant \ldots \geqslant \Psi_{d-1}(\mathfrak{T},\Lambda) = \Psi_1(-\mathfrak{T},\Lambda^*)$

Choosing Λ and $\mathfrak{T}$ — $d = n+1$

$$\Lambda = \begin{pmatrix} \mathbf{I}_1 & \\ -\Theta & \mathbf{I}_n \end{pmatrix} \mathbb{Z}^{n+1}, \qquad \mathfrak{T}: \quad \tau_1(s) = s, \quad \tau_2(s) = \ldots = \tau_{n+1}(s) = -s/n$$

Parametric geometry of numbers: Khintchine's transference theorem

Inequalities for the exponents

- $\Psi_1(\mathfrak{T},\Lambda) \leqslant \ldots \leqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{k} \leqslant \ldots \leqslant \dfrac{\Psi_{d-1}(\mathfrak{T},\Lambda)}{d-1} = \dfrac{\Psi_1(-\mathfrak{T},\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T},\Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{\Psi_k(\mathfrak{T},\Lambda)}{d-k} \geqslant \ldots \geqslant \Psi_{d-1}(\mathfrak{T},\Lambda) = \Psi_1(-\mathfrak{T},\Lambda^*)$

Choosing Λ and $\mathfrak{T}$ — $d = n+1$

$$\Lambda = \begin{pmatrix} \mathbf{I}_1 & \\ -\Theta & \mathbf{I}_n \end{pmatrix} \mathbb{Z}^{n+1}, \qquad \mathfrak{T}: \quad \tau_1(s) = s, \quad \tau_2(s) = \ldots = \tau_{n+1}(s) = -s/n$$

Khintchine's theorem — Parametric version

- $\Psi_1(\mathfrak{T},\Lambda) \leqslant \dfrac{\Psi_1(-\mathfrak{T},\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T},\Lambda)}{d-1} \geqslant \Psi_1(-\mathfrak{T},\Lambda^*)$

Intermediate Diophantine exponents

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad d = n + 1$$

k^{th} intermediate Diophantine exponent of Θ

$$\omega_k(\Theta) = \sup \left\{ \gamma \in \mathbb{R} \;\middle|\; |\mathbf{L} \wedge \mathbf{Z}| \leqslant |\mathbf{Z}|^{-\gamma} \text{ for infinitely many decomposable } \mathbf{Z} \in \textstyle\bigwedge^k(\mathbb{Z}^{n+1}) \right\}$$

Intermediate Diophantine exponents

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad d = n+1$$

k^{th} intermediate Diophantine exponent of Θ

$$\omega_k(\Theta) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, |\mathbf{L} \wedge \mathbf{Z}| \leqslant |\mathbf{Z}|^{-\gamma} \text{ for infinitely many decomposable } \mathbf{Z} \in \textstyle\bigwedge^k(\mathbb{Z}^{n+1})\right\}$$

$$\omega_1(\Theta) = \omega(\Theta), \quad \omega_n(\Theta) = \omega(\Theta^\intercal)$$

Laurent's theorem (2007)

$$(n-k)\omega_{k+1}(\Theta) \geqslant 1 + (n+1-k)\omega_k(\Theta)$$
$$k\omega_k(\Theta)^{-1} \leqslant 1 + (k+1)\omega_{k+1}(\Theta)^{-1}$$
$$k = 1, \ldots, n-1$$

Intermediate Diophantine exponents

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad d = n+1$$

k^{th} intermediate Diophantine exponent of Θ

$$\omega_k(\Theta) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, |\mathbf{L} \wedge \mathbf{Z}| \leqslant |\mathbf{Z}|^{-\gamma} \text{ for infinitely many decomposable } \mathbf{Z} \in \textstyle\bigwedge^k(\mathbb{Z}^{n+1})\right\}$$

$$\omega_1(\Theta) = \omega(\Theta), \quad \omega_n(\Theta) = \omega(\Theta^\intercal)$$

Laurent splits Khintchine

$$\begin{aligned} (n-k)\omega_{k+1}(\Theta) &\geqslant 1 + (n+1-k)\omega_k(\Theta) \\ k\omega_k(\Theta)^{-1} &\leqslant 1 + (k+1)\omega_{k+1}(\Theta)^{-1} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \omega(\Theta^\intercal) &\geqslant n - 1 + n\omega(\Theta) \\ \omega(\Theta)^{-1} &\leqslant n - 1 + n\omega(\Theta^\intercal)^{-1} \end{aligned}$$

Diophantine exponents of lattices

Diophantine exponent of Λ

$$\omega(\Lambda) = \sup\left\{\gamma \in \mathbb{R} \,\middle|\, \Pi(\mathbf{x}) \leqslant |\mathbf{x}|^{-\gamma} \text{ for infinitely many } \mathbf{x} \in \Lambda\right\}, \qquad \Pi(\mathbf{x}) = |x_1 \dots x_d|^{1/d}$$

Diophantine exponents of lattices

Diophantine exponent of Λ

$$\omega(\Lambda) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; \Pi(\mathbf{x}) \leqslant |\mathbf{x}|^{-\gamma} \text{ for infinitely many } \mathbf{x} \in \Lambda\right\}, \qquad \Pi(\mathbf{x}) = |x_1 \dots x_d|^{1/d}$$

Transference theorem (O.G., 2016)

$$1 + \omega(\Lambda)^{-1} \leqslant (d-1)^2\big(1 + \omega(\Lambda^*)^{-1}\big)$$

Diophantine exponents of lattices

Diophantine exponent of Λ

$\omega(\Lambda) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; \Pi(\mathbf{x}) \leqslant |\mathbf{x}|^{-\gamma} \text{ for infinitely many } \mathbf{x} \in \Lambda\right\}$, $\qquad \Pi(\mathbf{x}) = |x_1 \dots x_d|^{1/d}$

Transference theorem (O.G., 2016)

$$1 + \omega(\Lambda)^{-1} \leqslant (d-1)^2 \big(1 + \omega(\Lambda^*)^{-1}\big)$$

Schmidt–Summerer exponents

- space of parameters $\quad \mathcal{T} = \left\{\boldsymbol{\tau} = (\tau_1, \dots, \tau_d) \in \mathbb{R}^d \;\middle|\; \tau_1 + \ldots + \tau_d = 0\right\}$
- norm-like functional $\quad |\boldsymbol{\tau}|_+ = \max_{1 \leqslant k \leqslant d} \tau_k$
- Schmidt–Summerer exponents $\quad \Psi_k(\Lambda) = \liminf\limits_{|\boldsymbol{\tau}|_+ \to \infty} \dfrac{S_k(\boldsymbol{\tau})}{|\boldsymbol{\tau}|_+}$

Diophantine exponents of lattices

Diophantine exponent of Λ

$$\omega(\Lambda)=\sup\left\{\gamma\in\mathbb{R}\ \middle|\ \Pi(\mathbf{x})\leqslant|\mathbf{x}|^{-\gamma}\text{ for infinitely many }\mathbf{x}\in\Lambda\right\},\qquad \Pi(\mathbf{x})=|x_1\ldots x_d|^{1/d}$$

Transference theorem (O.G., 2016)

$$1+\omega(\Lambda)^{-1}\leqslant(d-1)^2\big(1+\omega(\Lambda^*)^{-1}\big)$$

Schmidt–Summerer exponents

- space of parameters $\mathcal{T}=\left\{\boldsymbol{\tau}=(\tau_1,\ldots,\tau_d)\in\mathbb{R}^d\ \middle|\ \tau_1+\ldots+\tau_d=0\right\}$
- norm-like functional $|\boldsymbol{\tau}|_+=\max_{1\leqslant k\leqslant d}\tau_k$
- Schmidt–Summerer exponents $\Psi_k(\Lambda)=\liminf_{|\boldsymbol{\tau}|_+\to\infty}\dfrac{S_k(\boldsymbol{\tau})}{|\boldsymbol{\tau}|_+}$

Transference theorem — Parametric version

$$\Psi_1(\Lambda)\leqslant\frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Schmidt–Summerer exponents

- space of parameters $\mathcal{T} = \left\{ \boldsymbol{\tau} = (\tau_1, \ldots, \tau_d) \in \mathbb{R}^d \,\middle|\, \tau_1 + \ldots + \tau_d = 0 \right\}$
- norm-like functional $|\boldsymbol{\tau}|_+ = \max_{1 \leqslant k \leqslant d} \tau_k$
- Schmidt–Summerer exponents $\Psi_k(\Lambda) = \liminf\limits_{|\boldsymbol{\tau}|_+ \to \infty} \dfrac{S_k(\boldsymbol{\tau})}{|\boldsymbol{\tau}|_+}$

Transference theorem — Parametric version

$$\Psi_1(\Lambda) \leqslant \frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Diophantine exponents of lattices

Schmidt–Summerer exponents

- space of parameters $\mathcal{T}=\left\{\boldsymbol{\tau}=(\tau_1,\ldots,\tau_d)\in\mathbb{R}^d \,\middle|\, \tau_1+\ldots+\tau_d=0\right\}$
- norm-like functional $|\boldsymbol{\tau}|_+=\max_{1\leqslant k\leqslant d}\tau_k$
- Schmidt–Summerer exponents $\Psi_k(\Lambda)=\liminf_{|\boldsymbol{\tau}|_+\to\infty}\dfrac{S_k(\boldsymbol{\tau})}{|\boldsymbol{\tau}|_+}$

Transference theorem — Parametric version

$$\Psi_1(\Lambda)\leqslant\frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Transference theorem split — Parametric version

$$\Psi_1(\Lambda)\leqslant\ldots\leqslant\frac{\Psi_k(\Lambda)}{k}\leqslant\ldots\leqslant\frac{\Psi_{d-1}(\Lambda)}{d-1}\leqslant\frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Diophantine exponents of lattices

Schmidt–Summerer exponents

- space of parameters $\mathcal{T}=\Big\{\boldsymbol{\tau}=(\tau_1,\ldots,\tau_d)\in\mathbb{R}^d\ \Big|\ \tau_1+\ldots+\tau_d=0\Big\}$
- norm-like functional $|\boldsymbol{\tau}|_+=\max_{1\leqslant k\leqslant d}\tau_k$
- Schmidt–Summerer exponents $\Psi_k(\Lambda)=\liminf\limits_{|\boldsymbol{\tau}|_+\to\infty}\dfrac{S_k(\boldsymbol{\tau})}{|\boldsymbol{\tau}|_+}$

Transference theorem — Parametric version

$$\Psi_1(\Lambda)\leqslant\frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Transference theorem split — Parametric version

$$\Psi_1(\Lambda)\leqslant\ldots\leqslant\frac{\Psi_k(\Lambda)}{k}\leqslant\ldots\leqslant\frac{\Psi_{d-1}(\Lambda)}{d-1}\leqslant\frac{\Psi_1(\Lambda^*)}{(d-1)^2}$$

Transference theorem split — Non-parametric version

$$1+\omega(\Lambda)^{-1}\leqslant\ldots\leqslant\ ???\ \leqslant\ldots\leqslant(d-1)^2\big(1+\omega(\Lambda^*)^{-1}\big)$$

Diophantine approximation with weights

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \dots \theta_n), \qquad \begin{array}{l} \boldsymbol{\rho} = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n \\ \rho_1 \geqslant \dots \geqslant \rho_n > 0 \\ \rho_1 + \dots + \rho_n = 1 \end{array}, \qquad |\mathbf{y}|_{\boldsymbol{\rho}} = \max_{1 \leqslant i \leqslant n} |y_i|^{1/\rho_i}$$

Diophantine approximation with weights

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad \begin{array}{l} \boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n \\ \rho_1 \geqslant \ldots \geqslant \rho_n > 0 \\ \rho_1 + \ldots + \rho_n = 1 \end{array}, \qquad |\mathbf{y}|_{\boldsymbol{\rho}} = \max_{1 \leqslant i \leqslant n} |y_i|^{1/\rho_i}$$

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta) = \sup\left\{ \gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0 \right\}$$

Weighted Diophantine exponent for $\Theta^\intercal$

$$\omega_{\boldsymbol{\rho}}(\Theta^\intercal) = \sup\left\{ \gamma \in \mathbb{R} \;\middle|\; |x + \Theta^\intercal \mathbf{y}| \leqslant |\mathbf{y}|_{\boldsymbol{\rho}}^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ \mathbf{y} \neq \mathbf{0} \right\}$$

Diophantine approximation with weights

$$\Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \Theta^\intercal = (\theta_1 \ldots \theta_n), \qquad \begin{matrix} \boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n \\ \rho_1 \geqslant \ldots \geqslant \rho_n > 0 \\ \rho_1 + \ldots + \rho_n = 1 \end{matrix}, \qquad |\mathbf{y}|_{\boldsymbol{\rho}} = \max_{1 \leqslant i \leqslant n} |y_i|^{1/\rho_i}$$

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta) = \sup \Big\{ \gamma \in \mathbb{R} \ \Big| \ |\Theta x - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0 \Big\}$$

Weighted Diophantine exponent for $\Theta^\intercal$

$$\omega_{\boldsymbol{\rho}}(\Theta^\intercal) = \sup \Big\{ \gamma \in \mathbb{R} \ \Big| \ |x + \Theta^\intercal \mathbf{y}| \leqslant |\mathbf{y}|_{\boldsymbol{\rho}}^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ \mathbf{y} \neq \mathbf{0} \Big\}$$

Transference theorem (O.G., 2019)

$$\omega_{\boldsymbol{\rho}}(\Theta^\intercal) \geqslant 1 - \rho_n + \rho_n \omega_{\boldsymbol{\rho}}(\Theta)$$

$$\omega_{\boldsymbol{\rho}}(\Theta)^{-1} \leqslant 1 - \rho_n + \rho_n \omega_{\boldsymbol{\rho}}(\Theta^\intercal)^{-1}$$

Diophantine approximation with weights: parametric approach

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0\right\}$$

Diophantine approximation with weights: parametric approach

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0\right\}$$

Choosing Λ and $\mathfrak{T}$ — $d = n + 1$

$$\Lambda = \begin{pmatrix} \mathbf{I}_1 & \\ -\Theta & \mathbf{I}_n \end{pmatrix} \mathbb{Z}^{n+1}, \qquad \mathfrak{T} = \mathfrak{T}_\gamma : \begin{array}{l} \tau_1(s) = s \\ \tau_{1+i}(s) = -\frac{1+((n+1)\rho_i-1)\gamma}{n+\gamma} s,\ i=1,\ldots,n \end{array}$$

Diophantine approximation with weights: parametric approach

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta)=\sup\left\{\gamma\in\mathbb{R}\ \middle|\ |\Theta x-\mathbf{y}|_{\boldsymbol{\rho}}\leqslant|x|^{-\gamma}\text{ admits }\infty\text{ solutions in }(x,\mathbf{y})\in\mathbb{Z}^{n+1},\ x\neq0\right\}$$

Choosing Λ and $\mathfrak{T}$ $\quad d=n+1$

$$\Lambda=\begin{pmatrix}\mathbf{I}_1 & \\ -\Theta & \mathbf{I}_n\end{pmatrix}\mathbb{Z}^{n+1},\qquad \mathfrak{T}=\mathfrak{T}_\gamma:\ \begin{aligned}&\tau_1(s)=s\\&\tau_{1+i}(s)=-\tfrac{1+((n+1)\rho_i-1)\gamma}{n+\gamma}s,\ i=1,\ldots,n\end{aligned}$$

"The best" of all γ

$$\Psi_1(\mathfrak{T}_\gamma,\Lambda)=\frac{1-\gamma}{d}\iff\omega_{\boldsymbol{\rho}}(\Theta)=\gamma$$

Diophantine approximation with weights: parametric approach

Weighted Diophantine exponent for Θ

$$\omega_{\boldsymbol{\rho}}(\Theta) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; |\Theta x - \mathbf{y}|_{\boldsymbol{\rho}} \leqslant |x|^{-\gamma} \text{ admits } \infty \text{ solutions in } (x, \mathbf{y}) \in \mathbb{Z}^{n+1},\ x \neq 0\right\}$$

Choosing Λ and $\mathfrak{T}$ — $d = n + 1$

$$\Lambda = \begin{pmatrix} \mathbf{I}_1 & \\ -\Theta & \mathbf{I}_n \end{pmatrix} \mathbb{Z}^{n+1}, \qquad \mathfrak{T} = \mathfrak{T}_\gamma : \begin{array}{l} \tau_1(s) = s \\ \tau_{1+i}(s) = -\frac{1+((n+1)\rho_i - 1)\gamma}{n+\gamma} s,\ i=1,\ldots,n \end{array}$$

"The best" of all γ

$$\Psi_1(\mathfrak{T}_\gamma, \Lambda) = \frac{1-\gamma}{d} \iff \omega_{\boldsymbol{\rho}}(\Theta) = \gamma$$

Transference theorem — Parametric version

- $\Psi_1(\mathfrak{T}_\gamma, \Lambda) \leqslant \dfrac{\Psi_1(-\mathfrak{T}_\gamma, \Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T}_\gamma, \Lambda)}{d-1} \geqslant \Psi_1(-\mathfrak{T}_\gamma, \Lambda^*)$

Diophantine approximation with weights: parametric approach

Transference theorem — **Parametric version**

- $$\Psi_1(\mathfrak{T}_\gamma,\Lambda)\leqslant\frac{\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)}{d-1}$$
- $$\frac{\Psi_1(\mathfrak{T}_\gamma,\Lambda)}{d-1}\geqslant\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)$$

Diophantine approximation with weights: parametric approach

Transference theorem — Parametric version

- $\Psi_1(\mathfrak{T}_\gamma,\Lambda)\leqslant\dfrac{\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T}_\gamma,\Lambda)}{d-1}\geqslant\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)$

Transference theorem split — Parametric version

- $\Psi_1(\mathfrak{T}_\gamma,\Lambda)\leqslant\ldots\leqslant\dfrac{\Psi_k(\mathfrak{T}_\gamma,\Lambda)}{k}\leqslant\ldots\leqslant\dfrac{\Psi_{d-1}(\mathfrak{T}_\gamma,\Lambda)}{d-1}=\dfrac{\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T}_\gamma,\Lambda)}{d-1}\geqslant\ldots\geqslant\dfrac{\Psi_k(\mathfrak{T}_\gamma,\Lambda)}{d-k}\geqslant\ldots\geqslant\Psi_{d-1}(\mathfrak{T}_\gamma,\Lambda)=\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)$

Transference theorem — Parametric version

- $\Psi_1(\mathfrak{T}_\gamma,\Lambda) \leqslant \dfrac{\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T}_\gamma,\Lambda)}{d-1} \geqslant \Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)$

Transference theorem split — Parametric version

- $\Psi_1(\mathfrak{T}_\gamma,\Lambda) \leqslant \ldots \leqslant \dfrac{\Psi_k(\mathfrak{T}_\gamma,\Lambda)}{k} \leqslant \ldots \leqslant \dfrac{\Psi_{d-1}(\mathfrak{T}_\gamma,\Lambda)}{d-1} = \dfrac{\Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)}{d-1}$
- $\dfrac{\Psi_1(\mathfrak{T}_\gamma,\Lambda)}{d-1} \geqslant \ldots \geqslant \dfrac{\Psi_k(\mathfrak{T}_\gamma,\Lambda)}{d-k} \geqslant \ldots \geqslant \Psi_{d-1}(\mathfrak{T}_\gamma,\Lambda) = \Psi_1(-\mathfrak{T}_\gamma,\Lambda^*)$

Transference theorem split — Non-parametric version

$$\omega_{\boldsymbol{\rho}}(\Theta^\intercal) \geqslant \ldots \geqslant \ ??? \ \geqslant \ldots \geqslant 1-\rho_n+\rho_n\omega_{\boldsymbol{\rho}}(\Theta)$$
$$\omega_{\boldsymbol{\rho}}(\Theta)^{-1} \leqslant \ldots \leqslant \ ??? \ \leqslant \ldots \leqslant 1-\rho_n+\rho_n\omega_{\boldsymbol{\rho}}(\Theta^\intercal)^{-1}$$

Two problems

Define intermediate lattice exponents

$$1+\omega(\Lambda)^{-1}\leqslant\ldots\leqslant\ ???\ \leqslant\ldots\leqslant(d-1)^2\big(1+\omega(\Lambda^*)^{-1}\big)$$

Define intermediate weighted exponents

$$\omega_{\boldsymbol{\rho}}(\Theta^\intercal)-1\geqslant\ldots\geqslant\ ???\ \geqslant\ldots\geqslant\rho_n(\omega_{\boldsymbol{\rho}}(\Theta)-1)$$

$$\omega_{\boldsymbol{\rho}}(\Theta)^{-1}-1\leqslant\ldots\leqslant\ ???\ \leqslant\ldots\leqslant\rho_n(\omega_{\boldsymbol{\rho}}(\Theta^\intercal)^{-1}-1)$$

Thank you for watching!