The ε-t-Net Problem

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ε-nets

Definition (ε-nets)

Let $\varepsilon \in (0, 1)$ and let $H = (V, \mathcal{E})$ be a hypergraph. An **ε-net** for H is a set $N \subseteq V$ such that $N \cap e \neq \emptyset$ for every $e \in \mathcal{E}$ such that $|e| \geq \varepsilon |V|$.

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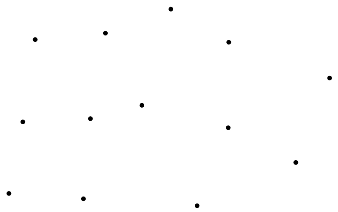

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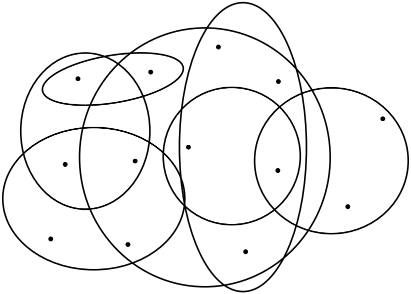

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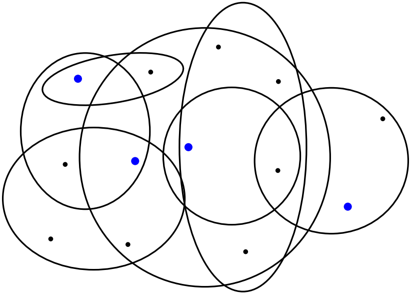

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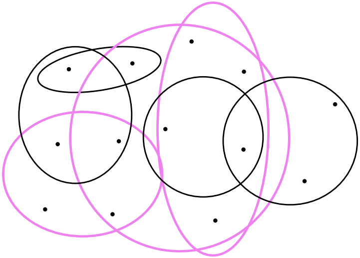

$|V| = 13$

$\varepsilon = \frac{2}{7}$

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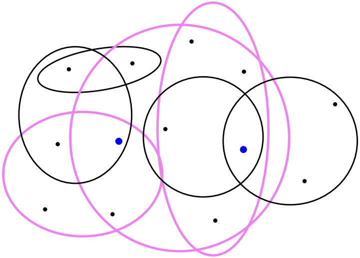

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The Goal: To find good bounds on ε-nets.

Bounds on ε-nets

Theorem (Haussler, Welzl '87)

All hypergraphs with VC-dimension d have ε-nets of size

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Theorem (Komlós, Pach, Woeginger '92)

All hypergraphs with VC-dimension d have ε-nets of size

$$O\left(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon}\right).$$

Moreover, there are hypergraphs for which the bound is tight.

VC-dimension

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Theorem (Sauer, Shelah '72)

Let $H = (V, \mathcal{E})$ be a hypergraph with VC-dimension d then for any $T \subseteq V$,

$$|\{T \cap e \mid e \in \mathcal{E}\}| = O(|T|^d).$$

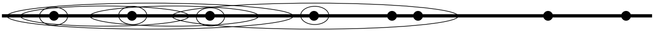
Geometric hypergraphs

Definition (intersection hypergraph)

Let B and R be two families of sets, the intersection hypergraph $H(B, R) = (V, \mathcal{E})$ is the hypergraph where $V = B$ and any $r \in R$ defines a hyperedge $\{b \in B : b \cap r \neq \emptyset\}$.

B : points on a line

R : a collection of intervals



$\mathrm{VC} - dim \leq 2$

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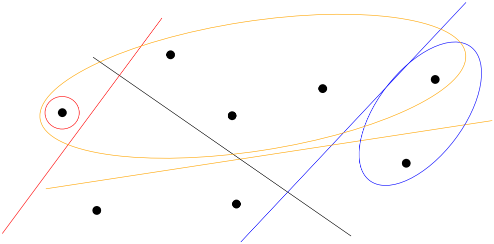
B : points in the plane

R : a collection of halfplanes



$\text{VC} - dim \leq 3$

ε-nets for geometric hypergraphs

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Upper Bounds:

- $H(P, \mathcal{H})$ where P is a set of points and $\mathcal{H}$ is a set of halfplanes in $\mathbb{R}^2$ (or halfspaces in $\mathbb{R}^3$) admits an ε-net of size $O\left(\frac{1}{\varepsilon}\right)$ (Matoušek, Seidel, Welzl '90; Pyrga, Ray '08; Har-Peled, Kaplan, Sharir, Smorodinsky '14).

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- $H(\mathcal{D}, P)$ where $\mathcal{D}$ is a collection of pseudo-disks and P is a set of points in $\mathbb{R}^2$ admits an ε-net of size $O\left(\frac{1}{\varepsilon}\right)$ (Pyrga, Ray '08).

ε-nets for geometric hypergraphs

Lower Bounds:

- $H(P, \mathcal{L})$ where P is a set of points and $\mathcal{L}$ is a set of lines in the plane can require an ε-net of size $\Omega\left(\frac{1}{\varepsilon}\sqrt{\frac{\log(1/\varepsilon)}{\log\log(1/\varepsilon)}}\right)$ (Balogh, Samotij '19).

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- $H(P, \mathcal{H})$ where P is a set of points and $\mathcal{H}$ is a set of hyperplanes in $\mathbb{R}^4$ can require an ε-net of size $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ (Pach, Tardos '10). Similar bound holds for higher dimensions (Kupavskii, Mustafa, Pach '16).

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Definition (ε-t-nets)

Let $H = (V, \mathcal{E})$ be a finite hypergraph, t a positive integer and $\varepsilon \in (t/|V|, 1)$. A family $N \subseteq \binom{V}{t}$ is an ***ε-t*-net** if for every $e \in \mathcal{E}$ with $|e| \geq \varepsilon |V|$ there is an $s \in N$ such that $s \subseteq e$.

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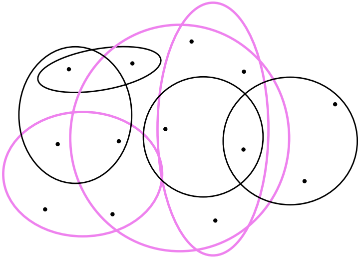

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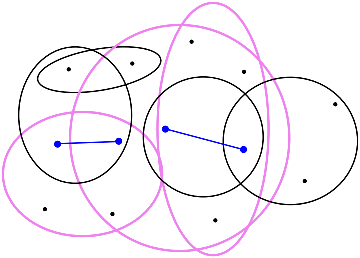

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$t = 2$

Previous work

For $t = \Theta(\varepsilon n)$, there are ε-t-nets of size $O_d\left(\frac{1}{\varepsilon^d}\right)$ (Mustafa, Ray '17; Dutta, Ghosh, Jartoux, Mustafa '19).

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Observation

The size of an ε-t-net is at least the size of an ε-net.

First attempt

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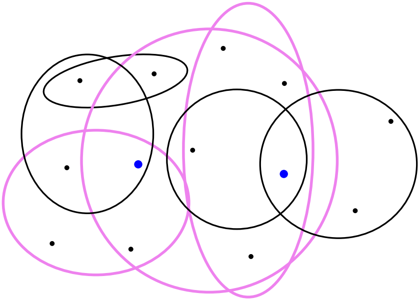

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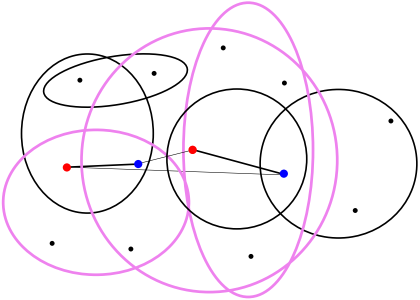

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We find an ε-2-net of size $|\varepsilon\text{-net}|^2 = O\left(\frac{d^2}{\varepsilon^2}\left(\log\frac{1}{\varepsilon}\right)^2\right)$.

Second attempt

For a hypergraph $H = (V, \mathcal{E})$, we define

$$H^t = \left(\binom{V}{t}, \mathcal{E}^t \right)$$

where

$$\mathcal{E}^t = \left\{ \binom{e}{t} \middle| e \in \mathcal{E} \right\}.$$

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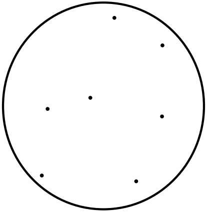

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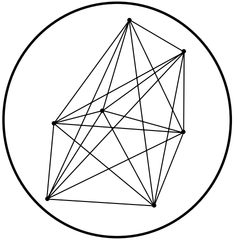

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Claim

Let H be a hypergraph with VC-dim$(H) = d$, then
$d - t + 1 \leq$ *VC-dim*$(H^t) \leq (2 \log 6t)d$.

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A ε^2-net for H^2 is an ε-2-net for H so H has an ε-2-net of size $O\left(\frac{d}{\varepsilon^2} \log \frac{1}{\varepsilon}\right)$.

Main result

Theorem (Alon, Jartoux, Keller, Smorodinsky, Y. '19)

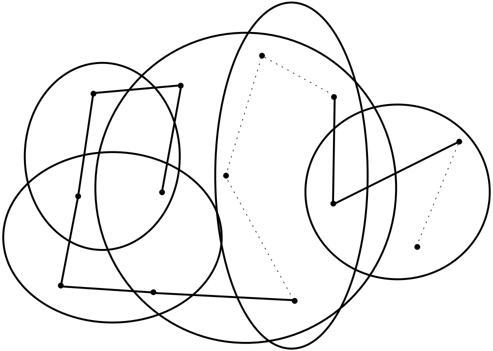
Let $\varepsilon \in (0,1)$ and $t \in \mathbb{N} \setminus \{0\}$. Then there exists a $C \in \mathbb{N}$ such that every hypergraph with VC-dimension d and on $> C\left(\frac{t-1}{\varepsilon}\right)^{2^d}$ vertices admits an ε-t-net of size $O(\frac{d(1+\log t)}{\varepsilon} \log \frac{1}{\varepsilon})$, all elements of which are pairwise disjoint.

Proof of the main result

Theorem (Welzl '88)

Let $H = (V, \mathcal{E})$ be a hypergraph on n vertices with VC-dimension d. Then there exists a permutation Π of V with at most $O(n^{1-\frac{1}{2^d}})$ crossings of any edge $e \in \mathcal{E}$.

$\Pi = (v_1, v_2, \ldots, v_n)$ is crossing an edge $e \in \mathcal{E}$ if there are vertices v_i, v_{i+1} such that $v_i \in e$ and $v_{i+1} \notin e$ or vice versa.



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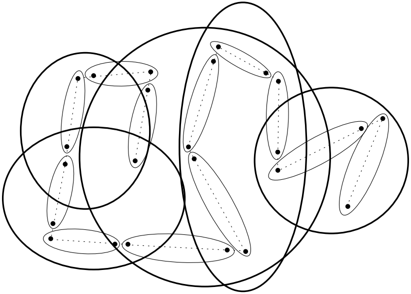

Let H^t_{lc} be a hypergraphs on
$V^t_{lc} = \big\{\{v_{kt}, v_{kt+1}, \ldots, v_{kt+t-1}\} : 0 \leq k < \lfloor \frac{|V|}{t} \rfloor\big\}$ and
$E^t_{lc} = \big\{\{v \in V^t_{lc} | v \subseteq e\} : e \in \mathcal{E}\big\}$.

Proof of the main result

Proof.

Let $H = (V, \mathcal{E})$, $|V| = n, \varepsilon > 0$, VC-$dim(H) = d$. Consider H^t_{lc}, then VC-$dim(H^t_{lc}) \leq (2 \log 6t)d$.

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Let N be a $\frac{\varepsilon}{2}$-net for H^t_{lc} of size $O(\frac{d}{\varepsilon} \log \frac{1}{\varepsilon})$.

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Let $H = (V, \mathcal{E})$, $|V| = n, \varepsilon > 0$, VC-$dim(H) = d$. Consider H_{lc}^t, then VC-$dim(H_{lc}^t) \leq (2 \log 6t)d$.

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Let N be a $\frac{\varepsilon}{2}$-net for H^t_{lc} of size $O(\frac{d}{\varepsilon} \log \frac{1}{\varepsilon})$. The set N is ε-t-net for H. Indeed, consider $e \in \mathcal{E}$ so $|e| \geq \varepsilon n$, then e contains at least $\frac{\varepsilon n}{t} - C' n^{1-\frac{1}{2^d}} \geq \frac{\varepsilon n}{2t}$ (for some $C' \in \mathbb{N}$) elements from V^t_{lc} and therefore a set from N. □

Open questions

- In which settings the existence of an ε-net of some order of magnitude, implies the existence of an ε-2-net of roughly the same order of magnitude?
- Is it possible to improve the lower bound on $|V|$? (We showed a lower bound which is exponential in $\frac{1}{\varepsilon}$.)

Thank you.