On equitable 2-partitions of Johnson graphs with the second eigenvalue

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Definitions

Let us denote $[n] = \{1, 2, \ldots, n\}$.

Let $J(n, w)$ (Johnson graph), $n \geq 2w$, be a graph, with the set of vertices $\binom{[n]}{w}$. Two vertices x, y are adjacent if $|x \cap y| = w - 1$.

Distance is defined as follows $d(x, y) = w - |x \cap y|$.

In other terms, vertices of this graph may be treated as binary vectors of length n containing exactly w ones. Two vertices are adjacent if corresponding vectors differ in exactly 2 coordinates.

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Examples

$J(n, 1)$ is the complete graph K_n.

$J(n, 2)$ is the line graph of K_n.

Definitions

The Johnson graph $J(n, w)$ is distance-regular of diameter w and has $w+1$ distinct eigenvalues $\lambda_i(n, w) = (w-i)(n-w-i) - i$, $i = 0, 1, \ldots w$.

Corresponding eigenspaces $V_i(n, w)$ have dimensions $\binom{n}{i} - \binom{n}{i-1}$, $i = 0, 1, \ldots w$.

For $v \in V_i(n, w)$ we have

$$Mv = \lambda_i(n, w)v,$$

where M is the adjacency matrix of $J(n, w)$.

Definitions

Let $G = (V, E)$ be a graph. A real–valued function $f : V \longrightarrow \mathbb{R}$ is called a λ–eigenfunction of G if the equality

$$\lambda \cdot f(x) = \sum_{y \in (x,y) \in E} f(y)$$

holds for any $x \in V$ and f is not the all-zero function. Note that the vector of values of a λ–eigenfunction is an eigenvector of the adjacency matrix of G with an eigenvalue λ. The support of a real–valued function f is the set of nonzeros of f.

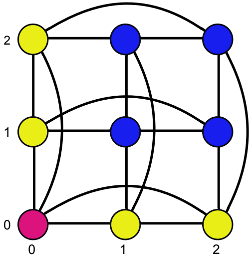
Equitable partitions

An r-partition $(C_1, C_2, \ldots, C_r)$ of the vertex set of a graph is called equitable with a quotient matrix $S = (s_{ij})_{i,j \in \{1,2,\ldots,r\}}$ if every vertex from C_i has exactly s_{ij} neighbours in C_j. The sets $C_1, C_2, \ldots, C_r$ are called cells of the partition.

Equitable partitions are also known as perfect colorings, regular partition and partition designs.

Example.

Equitable partition of 3×3–grid with a quotient matrix $\begin{pmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.



It is known (Cvetkovic, Doob, Sachs, 1980), that eigenvalues of M are eigenvalues of the adjacency matrix of the graph. By an eigenvalue of a partition we will understand an eigenvalue of its quotient matrix.

Codes and partitions

Let $G = (V, E)$ be a simple graph. An arbitrary subset $C \subseteq V$ is a code in G.

Equitable partitions include such classical objects as

- Perfect code C with radius r in a graph G:
  balls of radius r centred in vertices of C cover all vertices of the graph without intersections.
- Completely regular code
- Steiner system

Codes and partitions

Conjecture (Delsarte, 1973).
There are no non-trivial perfect codes in Johnson graphs.

This problem was considered by Bannai, Roos, Etzion, Schwarz and others. The most recent and strong results may be found in a series of papers by Etzion.

The conjecture is still open. Let us note, that Gordon in 2006 showed that there are no non-trivial 1-perfect codes in Johnson graphs $J(n, w)$ for $n \leq 2^{250}$.

Probably, investigation of equitable 2-partitions may give some ideas to find new approaches, at least for the case of perfect codes of radius 1.

Equitable 2-partitions $J(n, w)$

Let $C = (C_1, C_2)$ be an equitable partition of $J(n, w)$ with the quotient matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- $a + b = c + d = w(n - w)$.
- M has two eigenvalues: $w(n - w)$ and $a - c$.
- $|C_1| = \frac{c}{b+c}\binom{n}{w}$, $|C_2| = \frac{b}{b+c}\binom{n}{w}$.
- $b \geq c$.

For more complicated necessary conditions and known constructions we refer to a series of papers by Avgustinovich and Mogilnykh.

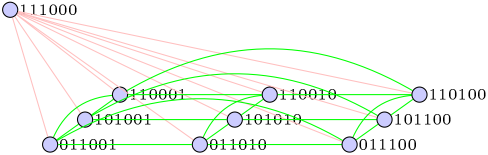
Equitable 2-partitions $J(n, w)$

Functions $f_1, f_2 : J(n, w) \to \mathbb{R}$ are equivalent if there exist a permutation $\pi \in S_n$ such that $\forall x \in J(n, w)$ we have $f_1(x) = f_2(\pi x)$. Two equitable 2-partitions (C_1, C_2) and (C_1', C_2') of the graph $J(n, w)$ are equivalent if the characteristic function χ_{C_1} is equivalent to $\chi_{C_1'}$ or $\chi_{C_2'}$.

Main problem

Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Is there an equitable 2-partition of a Johnson graph $J(n, w)$ with the quotient matrix M? What is the number of different and non-equivalent partitions with this matrix?

Example: $J(6,3)$



The Johnson graph $J(6,3)$ is the antipodal 9-regular graph on 20 vertices with diameter 3.

The set of neighbours of any vertex induces the 3×3 - grid.

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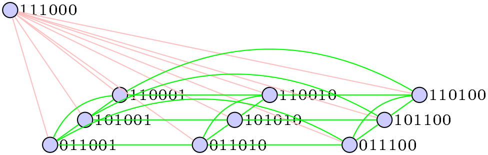

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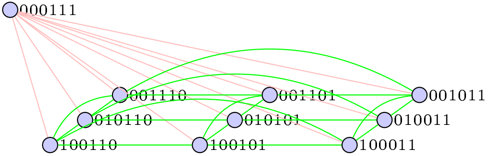

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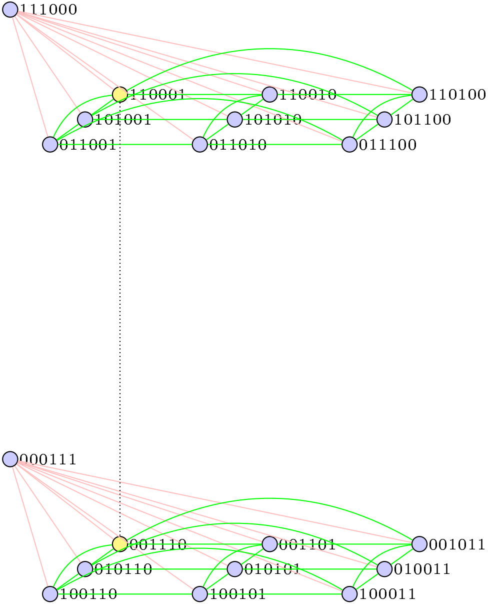

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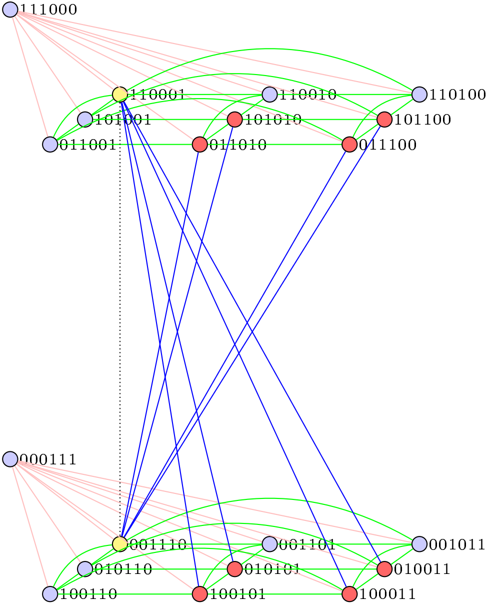

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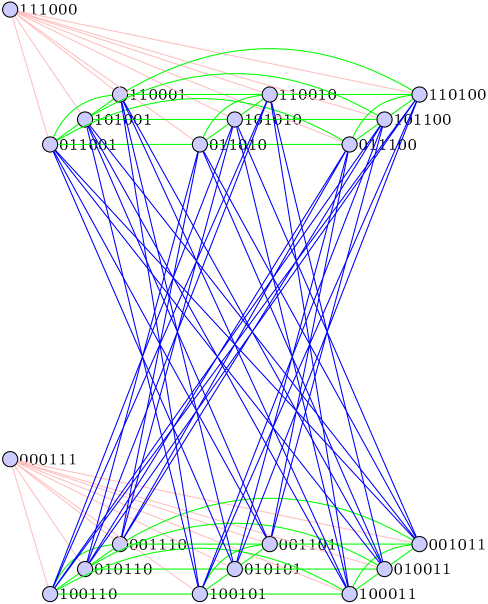

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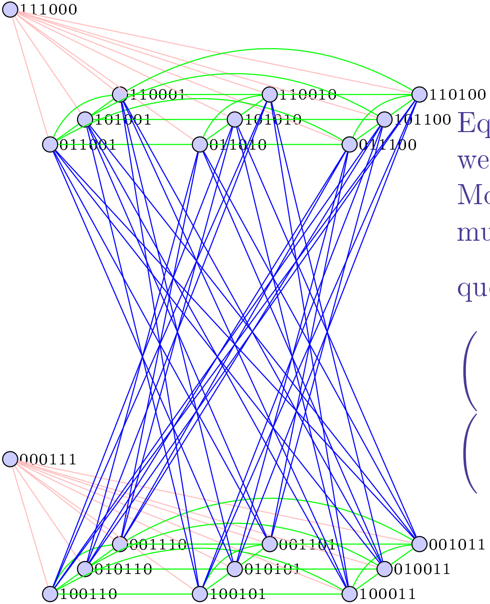
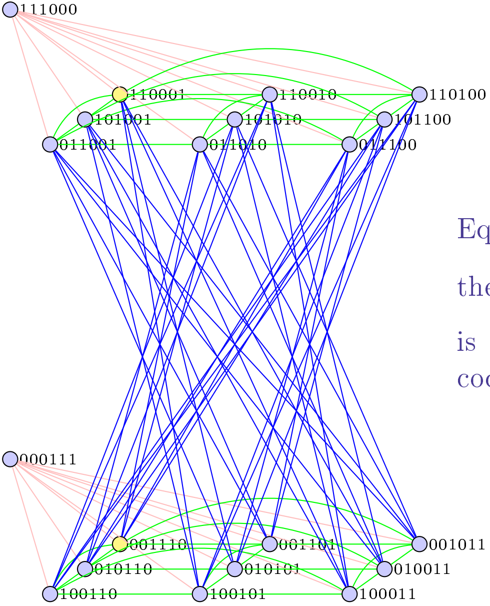
Equitable partitions of $J(6,3)$



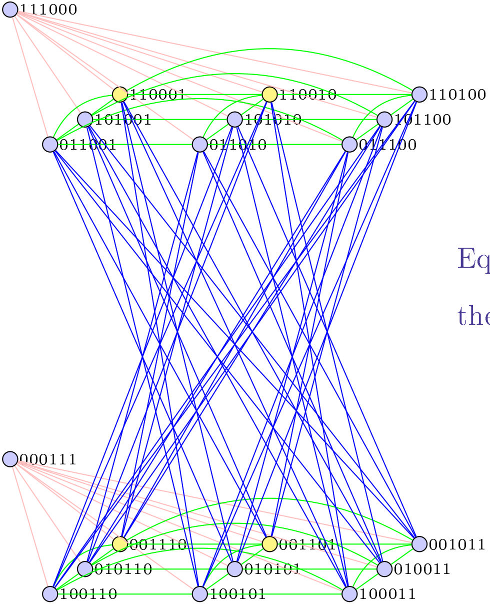
Equitable 2-partitions of $J(6,3)$ were listed by Avgustinovich and Mogilnykh in 2008. Such partitions must have one of the following quotient matrices: $\begin{pmatrix} 0 & 9 \\ 1 & 8 \end{pmatrix}$, $\begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}$, $\begin{pmatrix} 2 & 7 \\ 3 & 6 \end{pmatrix}$, $\begin{pmatrix} 3 & 6 \\ 4 & 5 \end{pmatrix}$, $\begin{pmatrix} 4 & 5 \\ 5 & 4 \end{pmatrix}$, $\begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}$, $\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$.

Equitable partitions of $J(6,3)$



Equitable 2-partition of $J(6,3)$ with the quotient matrix $\begin{pmatrix} 0 & 9 \\ 1 & 8 \end{pmatrix}$. This is an example of trivial 1-perfect code.

Equitable partitions of $J(6,3)$



Equitable 2-partition of $J(6,3)$ with the quotient matrix $\begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}$.

Equitable partitions of $J(n, w)$

One of approaches to characterisation of partitions is to fix their eigenvalues. In 2003 Meyerowitz described all equitable 2-partitions of Johnson graphs $J(n, w)$ and Hamming graphs $H(n, q)$ with the first eigenvalue ($\lambda_1(n, w)$ and $n(q-1)-q$ correspondingly).

In 2019 Mogilnykh and Valyuzhenich described all equitable 2-partitions of $H(n, q)$ with the second eigenvalue.

Let $C = (C_1, C_2)$ be an equitable partition of $J(n, w)$ with the quotient matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If M has the second eigenvalue $\lambda_2(n, w)$ in its spectra then $b + c = 2n - 2$.

There are partial results for $w = 3$ for partitions of Johnson graphs with the second eigenvalue.

Partitions of $J(n,3)$

Partitions of $J(n,3)$ were studied by Avgustinovich, Mogilnykh in a series of papers. In 2008 they listed all admissible quotient matrices for $J(6,3)$ and $J(7,3)$, and in 2010 - for $J(8,3)$. Nowadays, there are three known infinite series of equitable 2-partitions of a Johnson graph $J(2m,3)$ with the second eigenvalue with the following matrices:

- $\begin{pmatrix} 3(2m-5) & 6 \\ 4(m-2) & 2m-1 \end{pmatrix}$ and $\begin{pmatrix} 3(m-3) & 3m \\ m-2 & 5m-7 \end{pmatrix}$ (Godsil, Praeger, 1997).
- $\begin{pmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{pmatrix}$ (Avgustinovich, Mogilnykh, 2010).

Partitions of $J(n,3)$

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- $\begin{pmatrix} 3(m-1) & 3(m-2) \\ m+4 & 5m-13 \end{pmatrix}$ (Avgustinovich, Mogilnykh, 2010).

Theorem(Gavrilyuk, Goryainov, 2013).
There are no equitable partitions of $J(n,3)$ with the second eigenvalue for odd n.

Eigenspaces and partial differences

Given a real-valued $\lambda_i(n,w)$-eigenfunction f of $J(n,w)$ for some $i \in \{0,1,\dots,w\}$ and $j_1, j_2 \in \{1,2,\dots,n\}$, $j_1 < j_2$, define a a partial difference of f – a real-valued function f_{j_1,j_2} as follows: for any vertex $y = (y_1, y_2, \dots, y_{j_1-1}, y_{j_1+1}, \dots, y_{j_2-1}, y_{j_2+1}, \dots, y_n)$ of $J(n-2, w-1)$

$$f_{j_1,j_2}(y) = f(y_1, y_2, \dots, y_{j_1-1}, 1, y_{j_1+1}, \dots, y_{j_2-1}, 0, y_{j_2+1}, \dots, y_n)$$
$$-f(y_1, y_2, \dots, y_{j_1-1}, 0, y_{j_1+1}, \dots, y_{j_2-1}, 1, y_{j_2+1}, \dots, y_n).$$

Lemma(V., Mogilnykh, Valyuzhenich, 2018)
If f is a $\lambda_i(n,w)$-eigenfunction of $J(n,w)$ then f_{j_1,j_2} is a $\lambda_{i-1}(n-2, w-1)$-eigenfunction of $J(n-2, w-1)$ or the all-zero function.

$\{-1, 0, +1\}$-partial differences

Consider a characteristic function of one cell of some equitable 2-partition of $J(n, w)$ with the eigenvalue $\lambda_2(n, w)$ and take some partial difference of this function. The resulting function is a $\lambda_1(n-2, w-1)$-eigenfunction of $J(n-2, w-1)$ or the all-zero function.

In any case, this partial difference may take only three distinct values -1, 0, 1. As we see, the problem of constructing equitable 2-partition with $\lambda_2(n, w)$ may be reduced to the problem of constructing $\lambda_1(n-2, w-1)$-eigenfunctions with some restrictions.

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The following theorem gives a full classification of $\lambda_1(n-2, w-1)$-eigenfunctions we are interested in.

Theorem (V., 2020). If $f : J(n, w) \to \{-1, 0, 1\}$ is a $\lambda_1(n, w)$-eigenfunction of $J(n, w)$, $f \not\equiv 0$, $w \geq 2$, then f is equivalent up to multiplication by a non-zero constant to one of the following functions:

$\{-1, 0, +1\}$-partial differences

1. $$f_1(x) = \begin{cases} 1, x_1 = 1,\ x_2 = 0 \\ -1, x_1 = 0,\ x_2 = 1 \\ 0, & \text{otherwise.} \end{cases}$$
   $x = (x_1, x_2, \dots, x_n) \in J(n, w)$, $w \geq 2$ and $n \geq 2w$
2. $$f_2(x) = \begin{cases} 1, x_1 = 1,\ x_2 = 1 \\ -1, x_1 = 0,\ x_2 = 0 \\ 0, & \text{otherwise.} \end{cases}$$
   $x = (x_1, x_2, \dots, x_n) \in J(n, w)$, $w \geq 2$ and $n = 2w$.
3. $$f_3(x) = \begin{cases} 1, x_1 = 1, \\ -1, x_1 = 0, & x = (x_1, x_2, \dots, x_n) \in J(n, w), \\ 0, & \text{otherwise.} \end{cases}$$
   $w \geq 2$ and $n = 2w$.
4. $$f_4(x) = \begin{cases} 1, Supp(x) \subseteq \{1, 2, \dots, \frac{n}{2}\}, \\ -1, Supp(x) \subseteq \{\frac{n}{2}+1, \frac{n}{2}+2, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$
   $x = (x_1, x_2, \dots, x_n) \in J(n, w)$, $w = 2$, $n \geq 2w$ and n is even.

Partitions of $J(n, w)$, $n > 2w$

Theorem (V., 2020). There are no equitable 2-partitions in a Johnson graph $J(n, w)$, $n > 2w$, $w > 3$, with the quotient matrix

$$\begin{pmatrix} w(n-w)-b & b \\ 2n-2-b & w(n-w)-2n+2+b \end{pmatrix},$$

$b \in \{n-1, n, \ldots, 2n-1\}$.

Partitions of $J(n, w)$, $n > 2w$

Sketch of the proof.

- Suppose that there exists such an equitable 2-partition (C_1, C_2).
- Consider a function $f = b\chi_{C_1} - c\chi_{C_2}$. Clearly, f is $\lambda_2(n, w)$-eigenfunction of $J(n, w)$ and $f : J(n, w) \to \{b, -c\}$.
- Consider a function $g = \frac{1}{b+c} f_{i_1, i_2}$ defined on vertices of $J(n-2, w-1)$ for some i_1, i_2, $1 \leq i_1 < i_2 \leq n$, $g \not\equiv 0$. By definition, $g : J(n-2, w-1) \to \{-1, 0, +1\}$ and g is $\lambda_1(n-2, w-1)$-eigenfunction of $J(n-2, w-1)$. W.l.o.g. $i_1 = 1$, $i_2 = 2$.
- Using the characterization of such functions we have

$$g(\bar{x}) = \begin{cases} 1, x_3 = 1,\ x_4 = 0 \\ -1, x_3 = 0,\ x_4 = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$\bar{x} = (x_3, x_4, \ldots, x_n) \in J(n-2, w-1).$

Partitions of $J(n, w)$, $n > 2w$

Sketch of the proof.

- Therefore, we have the following equalities

$$f(1010\bar{z}) = f(0101\bar{z}) = b,\ \bar{z} \in J(n-4, w-2),$$

$$f(1001\bar{z}) = f(0110\bar{z}) = -c,\ \bar{z} \in J(n-4, w-2).$$

- Analysis of these equalities shows that $f_{i_1,i_2} \equiv 0$ for $i_1 \neq i_2$, $i_1, i_2 \in \{5, 6, \dots, n\}$.
- Therefore, our partition depends only on the first four coordinates.
- Further analysis of possible functions $\frac{1}{b+c} f_{i_1,i_2}$, $i_1, i_2 \in \{1, 2, 3, 4\}$ shows that n must be equal to $2w$.

Partitions of $J(n, w)$, $n > 2w$

Conjecture. For $i > 2$, there exists w_0 such that for all $w > w_0$ and $n > 2w$ there are no equitable partitions of $J(n, w)$ with the eigenvalue $\lambda_i(n, w)$.

Constructions for $J(2w, w)$

Construction 1 (Avgustinovich, Mogilnykh, 2010).
Let $C = (C_1, C_2)$ be a partition of the set of vertices of $J(2w, w)$, $w \geq 3$, defined by the following rule:
$C_1 = \{(x_1, x_2, x_3, \ldots, x_n) \in J(2w, w) | (x_1, x_2, x_3) \in \{(0, 0, 0), (1, 1, 1)\}\}$,
$C_2 = J(2w, w) \setminus C_1$.
Then $C = (C_1, C_2)$ is equitable with the quotient matrix

$$\begin{pmatrix} w^2 - 3w & 3w \\ w - 2 & w^2 - 2 + 2 \end{pmatrix}.$$

Constructions for $J(2w, w)$

Construction 2 (Martin, 1994).
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$C_1 = \{(x_1, x_2, \ldots, x_n) \in J(2w, w) | x_1 + x_2 = 0 \text{ or } 2\}$, $C_2 = J(2w, w) \setminus C_1$.
Then $C = (C_1, C_2)$ is equitable with the quotient matrix

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Constructions for $J(2w, w)$

Construction 3 (V., 2020).
Let $C = (C_1, C_2)$ be a partition of the set of vertices of $J(2w, w)$, $w \geq 5$, defined by the following rule:
$C_1 = \{(x_1, x_2, x_3, x_4, x_5, \ldots, x_n) \in J(2w, w) | (x_1, x_2, x_3, x_4, x_5) \in B\}$,
$C_2 = J(2w, w) \setminus C_1$, where
$B = \{(0,0,0,0,0), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1), (1,0,1,0,0), (0,1,0,1,0), (0,0,1,0,1), (0,0,0,1,1), (1,1,1,1,1), (1,1,0,1,1), (1,1,1,0,1), (1,1,1,1,0), (0,1,0,1,1), (1,0,1,0,1), (1,1,0,1,0), (1,1,1,0,0)\}$.
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$(0,1,1,1,1),(0,0,1,1,1),(0,1,0,1,1),(1,1,1,0,0)\}$.
Then $C = (C_1, C_2)$ is equitable with the quotient matrix

$$\begin{pmatrix} w^2 - 3w + 2 & 3w - 2 \\ w & w^2 - w \end{pmatrix}.$$

Partitions of $J(2w, w)$

Theorem (V., 2020).
Let $C = (C_1, C_2)$ be an equitable partition of $J(2w, w)$ with the second eigenvalue, $w \geq 7$. Then C is equivalent to one of the partitions from Constructions 1,2,3 and 4. For $w = 4$, $w = 5$ and $w = 6$ the set of admissible matrices is also covered by matrices from these Constructions.

The proof is also based on the analysis of possible partial differences. However, for $n = 2w$ there are three possible non-equivalent partial differences. Hence, the one need more accurate and deep analysis.

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Partitions of $J(2w, w)$

Idea of the proof. Similarly to the case $J(n, w)$, $n > 2w$, also uses possible partial differences. Besides that, it also uses so-called block partition of the set of coordinates $\{1, 2, \ldots, n\}$.

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Lemma 1 (V., Mogilnykh, Valyuzhenich, 2018). Let $f \in J(n, w) \to \mathbb{R}$. Let $f_{i_1,i_2} \equiv 0$ and $f_{i_1,i_3} \equiv 0$ for some pairwise distinct $i_1, i_2, i_3 \in \{1, 2, \ldots n\}$. Then $f_{i_2,i_3} \equiv 0$.

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By Lemma 1 the set of coordinate positions $\{1, 2, \ldots, n\}$ is partitioned into blocks. Let us denote by $BD(f)$ the set of these blocks. In other words, $\forall B \in BD(f) \forall i, j \in B$ such that $i \neq j$ we have $f_{i,j} \equiv 0$, and $\forall B, B' \in BD(f)$ such that $B \neq B'$ we have that $\forall i \in B \forall j \in B'$ $f_{i,j} \not\equiv 0$.

Partitions of $J(2w, w)$

Lemma 2 (V., 2020). Let $C = (C_1, C_2)$ be an equitable partition of $J(n, w)$ with the quotient matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\frac{bc}{b+c}\binom{n}{w} = \sum_{i,i|1\leq i\leq j\leq n} |S(f_{i,j})|,$$

where $S(f_{i,j})$ is a support for $f_{i,j}$.

Proof. Left side of the equality is just a number of edges connecting vertices from different cells. Since every edge of the graph appears exactly once in the sum from the right side we have the equality.

Conclusion

Main results:

- The characterization of equitable 2-partitions of Johnson graphs with the second eigenvalue was obtained for all graphs $J(n, w)$ except $J(n, 3)$, $J(12, 6)$, $J(10, 5)$, $J(8, 4)$. In particular, two new infinite series of partitions were found.
- A method of analysis of partial differences of partitions was developed.

Future directions:

- Complete remaining cases of partitions with the second eigenvalue.
- Generalize and apply developed methods for partitions with eigenvalues $\lambda_i(n, w)$, $i > 2$.

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Thank you for your attention!