An elementary approach to the operator method in additive combinatorics

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Additive energy

Let A, B be finite sets in any abelian group $\mathbf{G}$. Define the *additive energy* as

$$E(A,B) := \{a_1 - b_1 = a_2 - b_2 \ : \ a_1, a_2 \in A,\ b_1, b_2 \in B\}$$

$$E(A) := E(A,A) = \{a_1 - a_2 = a_3 - a_4 \ : \ a_i \in A,\ i = 1,2,3,4\}$$

$$E_3(A) := \{a_1 - a_2 = a_3 - a_4 = a_5 - a_6 \ : \ a_i \in A,\ i = 1,\ldots,6\}$$

We study extremal values of $E(A) := E(A,A)$ and $E_3(A)$. Note that

$$|A|^2 \le E(A) \le |A|^3$$

$$|A|^3 \le E_3(A) \le |A|^4$$

Geometric and multiplicative structures (generalizing of geometric progression)

Definition

A finite set $A = \{a_1 < \dots < a_n\} \subset \mathbb{R}$ is *convex* if

$$a_i - a_{i_1} < a_{i+1} - a_i, \; i = 2, \dots, n-1$$

Definition

Let $\mathbb{F}$ be a field. Say that a finite set $A \subset \mathbb{F}$ has *small product set* if $|A| \ll |AA|$

Conjecture(s) (Erdős, Harzheim, 1980 + Erdős, Szemerédi, 1983 + folklore)

Let $A \subset \mathbb{R}$ be a finite set. If A is convex or A has small product, then

$$E(A) \leq |A|^{2+o(1)}$$

Proved results

Theorem(s) (Iosevich, Konyagin, Rudnev, Ten, 2004 + Schoen, Shkredov, 2013)

Let $A \subset \mathbb{R}$ be a finite set. If A is convex or A has small product set, then

$$E(A, D) \ll |A||D|^{\frac{3}{2}}, \; E_3(A) = |A|^{3+o(1)}$$

Corollary (trivial, using $D = A$)

Let $A \subset \mathbb{R}$ be a finite set. If A is convex or A has small product set, then

$$E(A) \ll |A|^{\frac{5}{2}+o(1)}$$

Corollary (Shkredov, 2013, using "popular differences" $D \subset A - A$)

Let $A \subset \mathbb{R}$ be a finite set. If A is convex or A has small product set, then

$$E(A) \ll |A|^{\frac{32}{13}+o(1)} = |A|^{2.4615...+o(1)}$$

Main Shkredov's lemma

Extended Iverson notation

Let $[f(A)=0] := \sum_{a\in A}[f(a)=0]$, i. e. $[f(A)=0]$ is the number of solutions to

$$f(a)=0,\ \ a\in A$$

Lemma (Shkredov, 2012)

For any finite sets $A, D \in \mathbf{G}$ from any abelian group $\mathbf{G}$ we have

$$\begin{bmatrix} A-A=A-A=D \\ A-A=A-A=D \\ A-A=A-A=D \end{bmatrix} \le |A|^3 \sum_{x,y,z\in A} \begin{bmatrix} x-y=A-A \\ x-z=A-A=D \\ y-z=A-A=D \end{bmatrix}$$

Shkredov used the operator method. We provide an elementary proof.

The Cauchy – Schwarz inequality

Let f, g be any functions and A, B be any finite sets. The Cauchy–Schwarz inequality implies two important conclusions:

- "mirroring"

$$[f(A) = g(B)]^2 \leq [f(A) = f(A)][g(B) = g(B)]$$

- "copying"

$$[f(A,B) = 0]^2 \leq |A| \sum_{a \in A} \begin{bmatrix} f(a,B) = 0 \\ f(a,B) = 0 \end{bmatrix}$$

An elementary proof of Shkredov's lemma

For prove the Shkredov's lemma, we use "copying"

$$[A-A=A-A=D]^2 \le |A| \sum_{x \in A} \begin{bmatrix} x-A=A-A=D \\ x-A=A-A=D \end{bmatrix}$$

and apply "mirroring" (C. – S. ineq.) along diagonal (to summing by $z \in A, d \in D$)

$$\left(\sum_{z \in A, d \in D} \left(\sum_{x \in A} \begin{bmatrix} x-A=d \\ x-z=A-A=D \end{bmatrix} \right) [A-A=d] \right)^2 \le$$

$$\left(\sum_{z \in A,\ d \in D} \sum_{x,y \in A} \begin{bmatrix} x-A=y-A=D \\ x-z=A-A=D \\ y-z+A-A=D \end{bmatrix} \right) \left(|A| \sum_{d \in D} [A-A=d]^2 \right)$$

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An elementary proof of Shkredov's lemma

For prove the Shkredov's lemma, we use "copying"

$$[A - A = A - A = D]^2 \le |A| \sum_{x \in A} \begin{bmatrix} x - A = A - A = D \\ x - A = A - A = D \end{bmatrix}$$

and apply "mirroring" (C. – S. ineq.) along diagonal (to summing by $z \in A, d \in D$)

$$\left(\sum_{z \in A, d \in D} \left(\sum_{x \in A} \begin{bmatrix} x - A = d \\ x - z = A - A = D \end{bmatrix} \right) [A - A = d] \right)^2 \le$$

$$\left(\sum_{z \in A,\ d \in D} \sum_{x, y \in A} \begin{bmatrix} x - A = y - A = D \\ x - z = A - A = D \\ y - z + A - A = D \end{bmatrix} \right) \left(|A| \sum_{d \in D} [A - A = d]^2 \right)$$

An elementary proof of Shkredov's lemma

Hence

$$[A-A=A-A=D]^4 \leq$$

$$\leq [A-A=A-A=D]\cdot|A|^3 \sum_{x,y,z\in A}\begin{bmatrix} x-A=y-A \\ x-z=A-A=D \\ y-z=A-A=D \end{bmatrix}$$

Interpretation by graphs (case $D = A - A$)

Define the *convolutions multigraph* of a finite set A as $G_A = (A, E_A)$ where

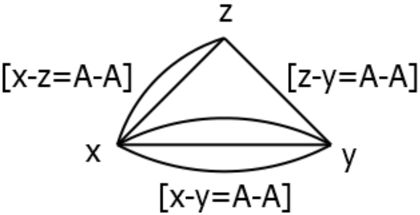
$$E_A(x, y) := (A \circ A)(x - y) = [x - y = A - A],\ x, y \in A$$

Notice that $\sum\limits_{x,y,z \in A} \begin{bmatrix} x - y = A - A \\ x - z = A - A \\ z - y = A - A \end{bmatrix}$ is the number of triangles in G_A

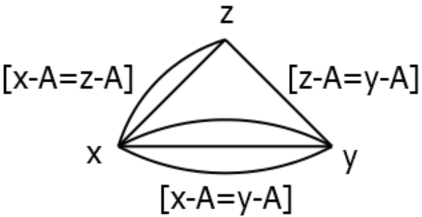
and $\sum\limits_{x \in A} \begin{bmatrix} x - A = A - A \\ x - A = A - A \end{bmatrix}$ is the number of "cherries" (P_2) in G_A.

We will illustrate transition (at the Cauchy – Schwarz inequality) from triangle to cherry by anti-mirroring.

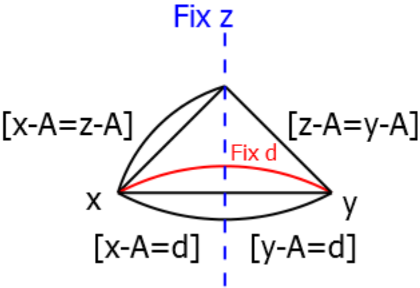
$$\sum_{x,y,z\in A}\begin{bmatrix} x-y=A-A \\ x-z=A-A \\ z-y=A-A \end{bmatrix}$$



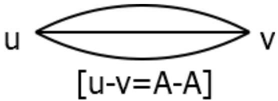
$$\sum_{x,y,z\in A}\begin{bmatrix} x-A=y-A \\ x-A=z-A \\ z-A=y-A \end{bmatrix}$$



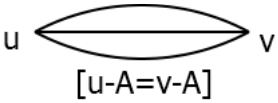
$$\sum_{x,y,z\in A}\begin{bmatrix} x-y=A-A \\ x-z=A-A \\ z-y=A-A \end{bmatrix} = \sum_{z\in A, d\in A-A}\left(\sum_{x\in A}\begin{bmatrix} x-A=d \\ x-z=A-A \end{bmatrix}\right)^2$$



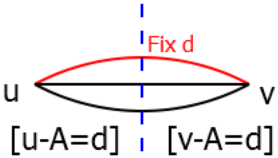
$$E(A) = \sum_{u,v \in A} [u - v = A - A]$$



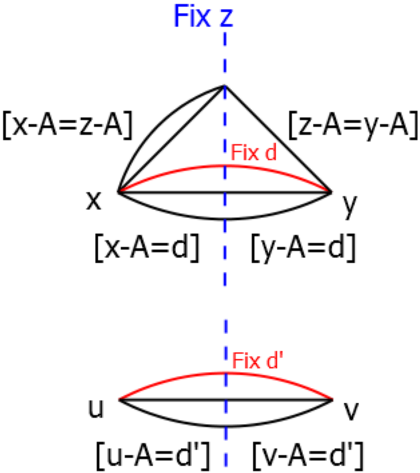
$$E(A) = \sum_{u,v \in A} [u - A = v - A]$$



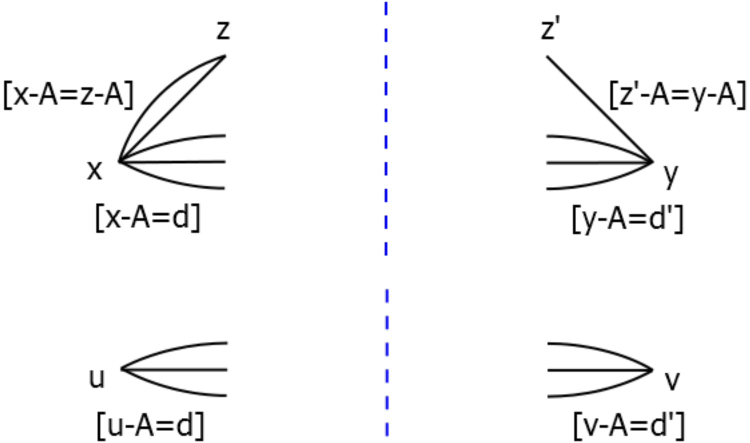
$$E(A) = \sum_{d \in A-A} \left(\sum_{u \in A} [u - A = d] \right)^2$$



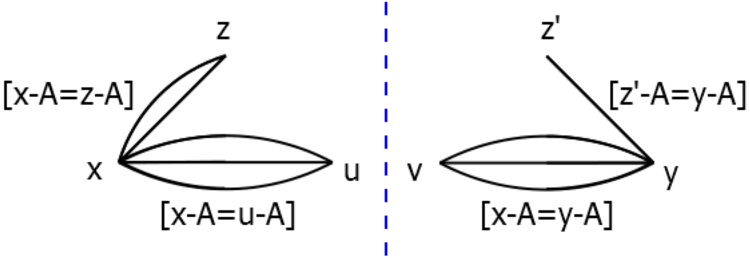
$$\left(\frac{1}{|A|}\sum_{\substack{z'\in A\\ d'\in A-A}}\left(\sum_{u\in A}[u-A=d]\right)^2\right)\left(\sum_{\substack{z\in A\\ d\in A-A}}\left(\sum_{x\in A}\begin{bmatrix} x-A=d \\ x-A=z-A\end{bmatrix}\right)^2\right)$$



$$\left(\sum_{z\in A, d\in A-A}\left(\sum_{u\in A}[A-u=d]\right)\left(\sum_{x\in A}\begin{bmatrix} x-A=d \\ x-A=z-A\end{bmatrix}\right)\right)^2$$



$$\left(\sum_{x,z,u\in A}\begin{bmatrix}x-z=A-A\\x-u=A-A\end{bmatrix}\right)^2=\left(\sum_{x\in A}\begin{bmatrix}x-A=A-A\\x-A=A-A\end{bmatrix}\right)^2$$



Sidorenko's conjecture

Definition

Let G be a multigraph and H be a graph. Denote

$$\mathcal{N}_H(G) := \sum_{\vec{x} \in V(G)^{V(H)}} \prod_{(v,u) \in E(H)} E_G(x_v, x_u)$$

Example

Let A be a finite set. If H is the triangle, then

$$\mathcal{N}_H(G_A) = \sum_{x,y,z \in A} \begin{bmatrix} x - y = A - A \\ y - z = A - A \\ z - x = A - A \end{bmatrix}$$

Sidorenko's conjecture

Sidorenko's conjecture $SC(G, H)$

Let H be a **bipartite** graph and G be any multigraph. Then

$$\frac{\mathcal{N}_H(G)}{|V(G)|^{|E(G)|}} \geq \left(\frac{\mathcal{N}_{K_2}(G)}{|V(G)|^2}\right)^{|V(H)|}$$

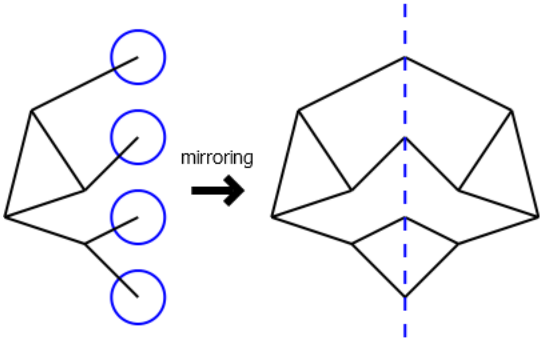
We state

Extended conjecture $EC(A, H)$

Let H be **any** graph and A be any finite set from some field. Then $SC(G_A, H)$ is true.

In particular, if H is the triangle then extended conjecture implies Shkredov's lemma for $D = A - A$.

Trivial approach to Sidorenko's conjecture is induction by mirroring.

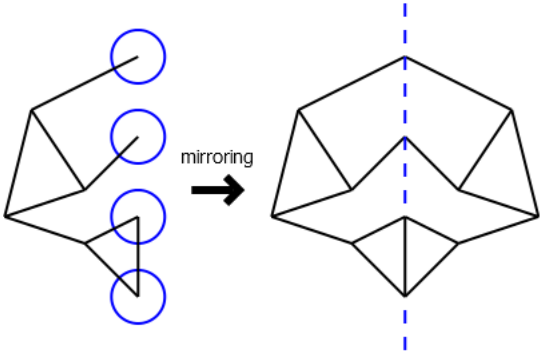


Let $H = (V, E)$ be the mirroring of $H' = (V', E')$ by vertexes $W = \{w_1, \dots, w_k\}$ and let W induce the empty subgraph. Then, applying the Cauchy-Swartz inequality for summing by $x_w,\ w \in W$, we see that

$$\mathcal{N}_{H'}(G)^2 \leq |G|^k \mathcal{N}_H(G)$$

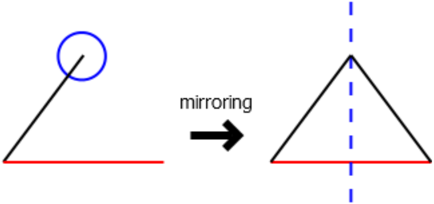
Hence, $SC(H', G) \Rightarrow SC(H, G)$.

Trivial approach to Sidorenko's conjecture is induction by mirroring.

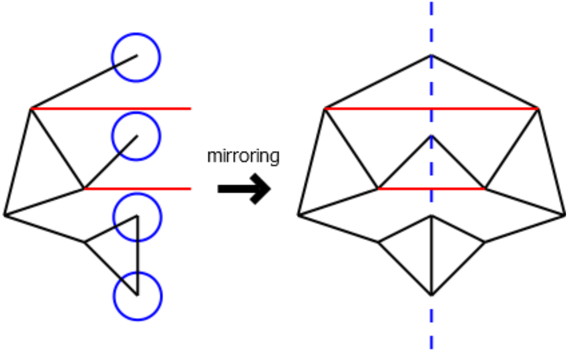


Similarly, if $H = (V, E)$ is the mirroring of $H' = (V', E')$ by vertexes $W = \{w_1, \ldots, w_k\}$ and W induce a matching subgraph, then $SC(G, H') \Rightarrow SC(G, H)$.

Let us return to Shkredov's lemma and the extended conjecture. Passing from cherry to triangle we have single edge that intersect "mirror line". It is allowed due to the specific structure of convolutions graphs.



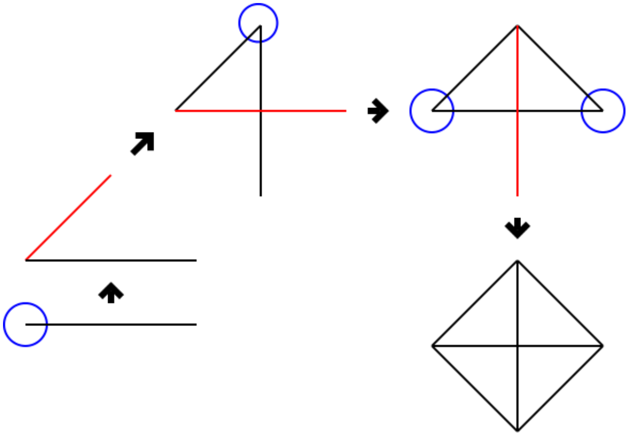
This principle can be generalize when all such edges is isolated (we call this "edged-mirroring")



Theorem

Let A be any finite set and H is the edged-mirroring of H'. Then $EC(A, H') \Rightarrow EC(A, H)$. Or, in other words, $SC(G_A, H') \Rightarrow SC(G_A, H)$

For example, we can build K_4 from K_2 by sequence of edged-mirrorings:



Corollary

Extended conjecture is true for $H = K_4$

Open questions:

- How to prove extended conjecture for some graphs H with arbitrary large chromatic number?
- How to generalize other methods for studying Sidorenko's conjecture to studying extended conjecture?
- How to generalize Shkredov's lemma with arbitrary D? (we need to describe edge subsets $E' \subset E$ for which we can add conditions $[x_u - x_v = D],\ (u, v) \in E'$ to right side of inequality)