String graphs have the Erdős-Hajnal property

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Erdős-Hajnal conjecture

- A family $\mathcal{G}$ of graphs has the **Erdős-Hajnal property**, if there exists $c > 0$ such that every $G \in \mathcal{G}$ contains a clique or an independent set of size $|V(G)|^c$.

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Erdős-Hajnal conjecture (1989)

Let H be a graph and let $\mathcal{G}$ be the family of graphs that do not contain H as an induced subgraph. Then $\mathcal{G}$ has the Erdős-Hajnal property.

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Erdős-Hajnal conjecture (1989)

Let $\mathcal{G}$ be a *hereditary* family of graphs that is not the family of all graphs. Then $\mathcal{G}$ has the Erdős-Hajnal property.

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The **intersection graph** of a family $\mathcal{F}$ of geometric objects is the graph whose vertices correspond to the elements of $\mathcal{F}$, and two vertices are joined by edge if the corresponding objects have a nonempty intersection.

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Is it also true that $\max\{\alpha(G), \omega(G)\} = \Omega(\sqrt{n})$?

Erdős-Hajnal property for intersection graphs

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- (Fox, Pach, Tóth 2011) The family of *intersection graphs of curves* such that any two curves intersect at most k times has the Erdős-Hajnal property.
- (Alon et al. 2005) A family of *semi-algebraic graphs* of bounded complexity has the Erdős-Hajnal property.

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Theorem (T. 2020+)

The conjecture is true.

Strong-Erdős-Hajnal property

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A family $\mathcal{G}$ of graphs has the **strong-Erdős-Hajnal property**, if there exists $c > 0$ such that for every $G \in \mathcal{G}$, either G or $\overline{G}$ contains a bi-clique of size $c|V(G)|$.

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- curves, any two intersect in at most k points (Fox, Pach, Tóth 2011)

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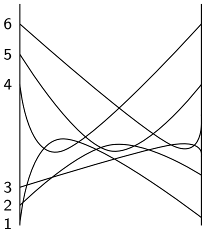

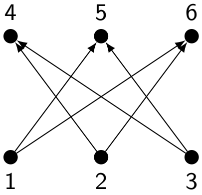

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- Every incomparability graph is a string graph (Lovász 1983).
- There exists an incomparability graph G on n vertices such that the largest bi-clique in G and $\overline{G}$ has size $O(n/\log n)$ (Fox 2006).

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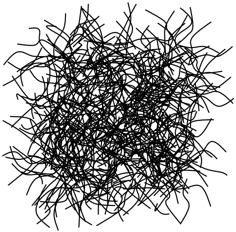

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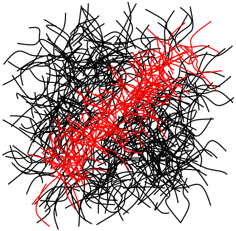

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- Therefore, the complement of every sparse string graph contains a linear sized bi-clique.

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Theorem

If G is a string graph on n vertices, then either G contains a bi-clique of size $\Omega(n/\log n)$, or the complement of G contains a bi-clique of size $\Omega(n)$.

Quasi-Erdős-Hajnal property

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A family of graphs $\mathcal{G}$ has the **quasi-Erdős-Hajnal property**, if there exists $c = c(\mathcal{G}) > 0$ such that the following holds. For every $G \in \mathcal{G}$ there exist t and t disjoint subsets $X_1, \ldots, X_t$ such that $t \geq (\frac{|V(G)|}{|X_i|})^c$, and either every X_i is complete to every X_j, or there are no edges between any X_i and X_j.

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Lemma (T. 2020+)

If G is a dense **incomparability graph** on n vertices, then there exist t and t disjoint sets $X_1, \dots, X_t$ such that $t \geq (\frac{|n|}{|X_i|})^c$, and X_i is complete to X_j.

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Theorem (T. 2020+)

The family of string graphs has the quasi-Erdős-Hajnal property.

Open questions

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What is the largest c such that every string graph on n vertices contains either a clique or an independent set of size $\Omega(n^c)$?

The best known upper bound is $c < 0.405$ (Kynčl 2012), which only uses segments.

Thank you for your attention!