Matrix Centralizers and their Applications

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Publications:

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Definition. For $A \in M_n(\mathbb{F})$ its *centralizer* $\mathcal{C}(A) = \{X \in M_n(\mathbb{F}) | \ AX = XA\}$ is the set of all matrices commuting with A.

Definition. For $S \subseteq M_n(\mathbb{F})$ its *centralizer* $\mathcal{C}(S) = \{X \in M_n(\mathbb{F}) | \ AX = XA$ for every $A \in S\}$ is the intersection of centralizers of all its elements.

Examples:

- $\mathcal{C}(I) = M_n(\mathbb{F})$
- $\mathcal{C}(E_{11}) = \{\alpha E_{11} \oplus M_{n-1}(\mathbb{F})\}$
- Let $\mathcal{D}_n(\mathbb{F})$ be diagonal matrices.

Then $\mathcal{C}(\mathcal{D}_n(\mathbb{F})) = \mathcal{D}_n(\mathbb{F})$.

- $\mathcal{C}(J_n(\lambda)) = \left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \dots & 0 & a_1 \end{pmatrix} | a_1, \dots, a_n \in \mathbb{F} \right\}$

- for $\lambda_1 \neq \lambda_2$ $\mathcal{C}(J_n(\lambda_1) \oplus J_k(\lambda_2)) =$

$$\left\{ \begin{pmatrix} a_1 & a_2 & \dots & a_n & 0 & \cdots & \cdots & 0 \\ 0 & a_1 & \ddots & \vdots & \vdots & & & \vdots \\ \vdots & \ddots & \ddots & a_2 & \vdots & & & \vdots \\ 0 & \dots & 0 & a_1 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & b_1 & b_2 & \dots & b_k \\ \vdots & & & \vdots & 0 & b_1 & \ddots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & b_2 \\ 0 & \cdots & \cdots & 0 & 0 & \dots & 0 & b_1 \end{pmatrix} | \begin{smallmatrix} a_1, & \dots, & a_n \\ b_1, & \dots, & b_k \end{smallmatrix} \in \mathbb{F} \right\}$$

- $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \oplus \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then

$$\mathcal{C}(A) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 & d_1 & d_2 \\ 0 & a_1 & a_2 & 0 & d_1 \\ 0 & 0 & a_1 & 0 & 0 \\ 0 & c_1 & c_2 & b_1 & b_2 \\ 0 & 0 & c_1 & 0 & b_1 \end{pmatrix} | a_i, b_i, c_i, d_i \in \mathbb{F} \right\}$$

$$\mathcal{C}_1(A) = \mathcal{C}(A)$$

$$\mathcal{C}_2(A) = \mathcal{C}(\mathcal{C}(A))$$

$$\mathcal{C}_{100}(A) = ?$$

Theorem

$\forall\ \mathbb{F}\ \forall\ A \in M_n(\mathbb{F})$ we have $\mathcal{C}_2(A) = \mathbb{F}[A]$,

here $\mathbb{F}[A]$ is the unital algebra over $\mathbb{F}$ generated by A.

Corollary.

1. $\mathcal{C}_1(A) = \mathcal{C}_3(A) = \mathcal{C}_5(A) = \ldots = \mathcal{C}_{2k-1}(A)$,
2. $\mathcal{C}_2(A) = \mathcal{C}_4(A) = \mathcal{C}_6(A) = \ldots = \mathcal{C}_{2k}(A) = \mathbb{F}[A]$.

Consequently, $\mathcal{C}_{100}(A) = \mathbb{F}[A]$

2 natural relations on $M_n(\mathbb{F})$ induced by centralizers:

Definition Preorder: $A \preceq B$ if $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ for $A, B \in M_n(\mathbb{F})$.

Definition $\mathcal{C}$-equivalence: $A \sim B$ if $\mathcal{C}(A) = \mathcal{C}(B)$.

Observation The preorder induces a partial order on a set of equivalence classes $M_n(\mathbb{F})/\sim$.

Proposition $A, B \in M_n(\mathbb{F})$. Then

1. $\mathcal{C}(A) \subseteq \mathcal{C}(B)$ iff $B \in \mathbb{F}[A]$.
2. $\mathcal{C}(A) = \mathcal{C}(B)$ iff $\mathbb{F}[A] = \mathbb{F}[B]$.

Proof. 1.

• $\mathcal{C}(A) \subseteq \mathcal{C}(B) \Rightarrow B \in \mathcal{C}(\mathcal{C}(B)) \subseteq \mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$.

• Conversely, if $B \in \mathbb{F}[A]$ then $B = p(A)$ for some $p \in \mathbb{F}[x]$, so if X commutes with A then it also commutes with $p(A) = B$.

Definition A is *minimal* if $\forall\ X \in M_n(\mathbb{F})$ with $\mathcal{C}(A) \supseteq \mathcal{C}(X)$ it follows that $\mathcal{C}(A) = \mathcal{C}(X)$.

Example 1. $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & n \end{pmatrix}$ is minimal.

2. $A = \lambda I$ is not minimal since $\mathcal{C}(X) \preceq \mathcal{C}(A)\ \forall\ X \in M_n(\mathbb{F}) \setminus \{\lambda X\}$.

Diagonal matrix with $\lambda_i \neq \lambda_j\ \forall\ i, j$ is minimal, since $\mathcal{C}(A) = \mathcal{D}_n(\mathbb{F})$, diagonal matrices.

Definition $A \neq \lambda I$ is *maximal* if $\forall\ X \in M_n(\mathbb{F}) \setminus \{\lambda I\}$ with $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ it follows that $\mathcal{C}(A) = \mathcal{C}(X)$.

$\mathcal{C}(\lambda I) = M_n(\mathbb{F})$

$A \neq \lambda I$: otherwise λIs are maximal, and only them

$X \neq \lambda I$: otherwise $\nexists$ maximal matrices

Definition $A \neq \lambda I$ is *maximal* if $\forall\ X \in M_n(\mathbb{F}),\ X \neq \lambda I$ with $\mathcal{C}(A) \subseteq \mathcal{C}(X)$ it follows that $\mathcal{C}(A) = \mathcal{C}(X)$.

Examples 1. $A = E_{ii}$ is maximal

2. $\forall$ idempotent matrix is maximal

Theorem. Let $A \in M_n(\mathbb{F})$. Then A is maximal iff $\forall$ $\lambda I \neq X \in \mathbb{F}[A]$ we have $\mathbb{F}[X] = \mathbb{F}[A]$.

Examples 1. $A = diag(1, 1, 2, \ldots, n-1) \in M_n(\mathbb{F})$, $n > 2$ is not maximal and not minimal.

2. $B = E_{11} \in M_2(\mathbb{F})$ is both minimal and maximal: B is diagonal with pairwise different entries and B is idempotent.

3. $C = E_{11} \in M_3(\mathbb{F})$ is maximal, but not minimal.

4. $D = diag(1, 2, \ldots, n) \in M_n(\mathbb{F})$ is minimal, but not maximal.

In what terms can we characterize Minimal and Maximal matrices?

1. Diameters of commuting graphs

2. Length function on matrices

3. Extremal commutative subalgebras

Diameters of commuting graphs

Definition Let S be a multiplicative algebraic structure. A *commuting graph* $\Gamma(S)$ of S is a simple graph:

- vertices are all non-central elements of S,
- $a \neq b$ incident to the same edge iff $ab = ba$.

Definition *Path* is

$$(A_0, A_k) = \{A_0, A_1, \ldots, A_k\colon\ A_i \neq A_j\ \forall\ 0 \leq i \neq j \leq k,\quad A_i,\ A_{i-1} \text{ are connected by an edge } \forall\ i = 1, \ldots, k\}.$$

k is the *length* of the path.

Definition The *distance* $d(A, B)$ is the length of the shortest path connecting A, B, $d(A, B) = \infty$ if $\nexists$ path A to B.

Definition Diameter $diam(\Gamma) = \max_{A,B \in v(\Gamma)} d(A, B)$.

Commutativity graph is not always connected.

Ex. 1 $M_2(\mathbb{F})$: a path from E_{11} to E_{12} does not exists since $\mathcal{C}(E_{11}) = \mathcal{D}_2(\mathbb{F})$ and

$\mathcal{C}(E_{12}) = \{aE_{12} + bE_{21} | a, b \in \mathbb{F}\}$,

$\mathcal{C}(E_{12}) \cap \mathcal{D}_2(\mathbb{F}) = \{0\}$ – scalar matrix

Ex. 2 A path from $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ to $B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$:

$$A \text{ — } E_{11} \text{ — } E_{22} \text{ — } E_{13} \text{ — } B$$

$d(A, B) = 4$

Extremal commutative subalgebras

What is the maximal number N of pair-wise commuting lin/ind operators in $M_n(\mathbb{F})$?

In other words, what do we know about the dimension of $\mathcal{A} \subseteq M_n(\mathbb{F})$ which is maximal commutative?

Theorem. [Schur, 1905] $\dim \mathcal{A} \leq [n^2/4] + 1$.

Theorem. [Laffey, 1985] $\dim \mathcal{A} > (2n)^{2/3} - 1$.

Length function

Let $\mathcal{A}$ be a finite dimensional associative algebra over $\mathbb{F}$, $\mathcal{S} = \{a_1, \ldots, a_k\}$ be its finite generating system

Definition *Length* of the word $a_{i_1} \ldots a_{i_t}$ where $a_{i_j} \in \mathcal{S}$ is t. 1 is a word of the length 0.

$\mathcal{L}_i(\mathcal{S})$ is the linear span of the words in $\mathcal{S}^i$. Note that $\mathcal{L}_0(\mathcal{S}) = \langle 1_{\mathcal{A}} \rangle = \mathbb{F}$ for unitary algebras, and $\mathcal{L}_0(\mathcal{S}) = 0$, otherwise. Let also $\mathcal{L}(\mathcal{S}) = \bigcup\limits_{i=0}^{\infty} \mathcal{L}_i(\mathcal{S})$.

Definition The *length of the generating system* $\mathcal{S}$ for the finite-dimensional algebra $\mathcal{A}$ is the number $l(\mathcal{S}) = \min\{k \in \mathbb{Z}_+ : \mathcal{L}_k(\mathcal{S}) = \mathcal{A}\}$.

Definition The *length of the algebra* $\mathcal{A}$ is defined to be the number $l(\mathcal{A}) = \max\{l(\mathcal{S}) : \mathcal{L}(\mathcal{S}) = \mathcal{A}\}$.

Theorem. [Paz, 1984] $l(M_n(\mathbb{F})) \leq \lceil (n^2+2)/3 \rceil$.

Theorem. [Pappacena, 1997]
$l(M_n(\mathbb{F})) < n\sqrt{2n^2/(n-1) + 1/4} + n/2 - 2$.

Conjecture. [Paz, 1984] Let $\mathbb{F}$ be an arbitrary field. Then $l(M_n(\mathbb{F})) = 2n - 2$.

It is true if $n \leq 4$.

Theorem. [Laffey, 1986] $\forall\ \mathbb{F}\ \exists$ a generating set $\mathcal{S} \subset M_n(\mathbb{F})$ such that $l(\mathcal{S}) = 2n - 2$.

Hence, $l(M_n(\mathbb{F})) \geq 2n - 2$.

Definition $A \in M_n(\mathbb{F})$ is *non-derogatory* if its minimal polynomial equals its characteristic polynomial.

Theorem. [Guterman, Laffey, Markova, Šmigoc, 2018]
$l(\mathcal{S}) \leq 2n - 2$ if $\mathcal{S}$ contains a non-derogatory matrix.

Thm. $\mathbb{F} = \overline{\mathbb{F}}$, $n \geq 3$. For $A \in M_n(\mathbb{F})$ TFAE.

(i) A is non-derogatory.

(ii) A is minimal. (iii) $\mathcal{C}(A) = \mathbb{F}[A]$.

(iv) $\mathbb{F}[A]$ is max commutative in $M_n(\mathbb{F})$ wrt inclusion.

(v) A is 1-regular (geom. multiplicity $=1$).

(vi) $J(A) = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, $\lambda_i \neq \lambda_j \ \forall \ i \neq j$.

(vii) $\exists \ v$: $v, Av, A^2v, \ldots, A^{n-1}v$ is basis of $\mathbb{F}^n$.

(viii) $\mathbb{F}[A]$ has maximal length among commutative subalgebras of $M_n(\mathbb{F})$.

(ix) $\exists \ X \in M_n(\mathbb{F})$: in a commuting graph $d(A, X) = 4$.

(x) A is freely integrable.

Definition $A \in M_n(\mathbb{C}), B \in M_{n+1}(\mathbb{C})$, then B is an integral of A, if $B = \begin{bmatrix} A & u \\ v^\top & \tau(A) \end{bmatrix}$, and also $\chi_A(x) = \frac{1}{n}\chi_B'(x)$. Then (u, v) is an integrator of A and $\det(B)$ is a constant of integration.

Example. Fix $b \in \mathbb{C} \setminus \{1\}$. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. Observe that $B_t = \begin{pmatrix} 1 & 0 & 1 \\ 0 & b & 1 \\ \frac{2t-3b+1}{2(1-b)} & \frac{(b-1)^3+2t-3b+1}{2(1-b)} & \frac{b+1}{2} \end{pmatrix}$ is an integral of A with the constant of integration t.

Definition

- A is integrable if $\exists$ its integral,
- A is uniquely integrable, if it is integrable and $\forall$ integrals B of A have the same determinant: $\exists\ \alpha \in \mathbb{C}$ s.t. $det(B) = \alpha\ \forall\ B$ which is an integral of A,
- A is freely integrable, if $\forall\ \alpha \in \mathbb{C}\ \exists$ an integral B of A s.t. $det(B) = \alpha$.

In Example A is freely integrable. Consider $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and write $B = \begin{pmatrix} 1 & 0 & u_1 \\ 0 & 1 & u_2 \\ v_1 & v_2 & 1 \end{pmatrix}$. We have $\chi_{A_1}(x) = x^3 - 3x^2 + (3 + u_1v_1 - u_2v_2)x + (u_1v_1 - u_2v_2 - 1)$

$\Longrightarrow 3 + u_1v_1 - u_2v_2 = 3$. It has solutions and

$det(A_1) = (u_1v_1 - u_2v_2) - 1 = -1$ $\forall$ solution

$\Longrightarrow$ B is uniquely integrable.

Thm. [Dolinar, Guterman, Kuzma, Oblak]

For $A \in M_n(\mathbb{F})$, $n \geq 3$, $|\mathbb{F}| \geq n+1$ TFAE

(i) A is non-derogatory.

(ii) A is minimal.

(iii) $\mathcal{C}(A) = \mathbb{F}[A]$.

(iv) $\mathbb{F}[A]$ is a max commutative in $M_n(\mathbb{F})$ wrt inclusion.

(v) A is 1-regular.

(vi) $J(A) = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$, $\lambda_i \neq \lambda_j \ \forall \ i \neq j$.

(vii) $\exists\ v$: $v, Av, A^2v, \ldots, A^{n-1}v$ is basis of $\mathbb{F}^n$.

(viii) $\mathbb{F}[A]$ has maximal length among all commutative subalgebras of $M_n(\mathbb{F})$.

Matrices with maximal centralizers

Theorem. [Dolinar, Šemrl, 2004]

$A \in M_n(\mathbb{C})$ is maximal iff it is $\mathcal{C}$-equivalent to an idempotent or a square-zero matrix $\neq O$, i.e.,

$$A \text{ is } \alpha I + \beta P \text{ or } \alpha I + \beta N,$$

where $\beta \neq 0$, I is the identity matrix,

$P^2 = P$, and $N^2 = 0$.

Theorem. [Dolinar, Guterman, Kuzma, Oblak, 2013]

$\mathbb{F}$ is arbitrary, $\lambda I \neq A \in M_n(\mathbb{F})$. TFAE

(i) A is maximal.

(ii) A belongs to one of the following three classes:

- A is $\mathcal{C}$-equivalent to an idempotent,
- A is $\mathcal{C}$-equivalent to a square-zero matrix,
- A is similar to $M \oplus \cdots \oplus M$, where M is a companion matrix of an irreducible polynomial, s.t. $\nexists$ a proper intermediate field between $\mathbb{F}$ and $\mathbb{F}[M]$.

APPLICATIONS

Commuting graphs

Theorem. [Akbari, Bidkhori, and Mohammadian, 2006]

1. $\mathbb{F} = \overline{\mathbb{F}}$, $n \geq 3$. Then $\Gamma(M_n(\mathbb{F}))$ is connected and $diam(\Gamma(M_n(\mathbb{F}))) = 4$.

2. $\Gamma(M_n(\mathbb{Q}))$ is disconnected for all n.

1: $A, B \in M_n(\mathbb{F})$ are given, $n \geq 3$.

- x is eigenv. of A corr. λ, y is eigenv. of A^t corr. λ,
- hence, A — $(x \cdot y^t)$ since

$$(A - \lambda I)(x \cdot y^t) = 0 = (x \cdot y^t)(A - \lambda I),$$

- f is eigenv. of B corr. μ, g is eigenv. of B^t corr. μ,
- hence, $(f \cdot g^t)$ — B
- $n \geq 3 \Rightarrow \exists\ z, h$ with $y^t z = 0 = g^t z$ and $h^t x = 0 = h^t f$
- THEN A — (xy^t) — (zh^t) — (fg^t) — B
- $d(J, J^t) = 4$.

Theorem. [Akbari, Bidkhori, and Mohammadian, 2008] $\Gamma(M_n(\mathbb{F}))$ is connected iff every field extension of $\mathbb{F}$ of degree n contains ≥ 1 proper intermediate field.

Theorem. [Dolinar, Guterman, Kuzma, Oblak, 2014] Let $n \geq 2$ and let $\mathbb{F}$ be an arbitrary field. A commuting graph $\Gamma(M_n(\mathbb{F}))$ is not connected iff $\exists\ A \in M_n(\mathbb{F})$ which is simultaneously minimal and maximal.

Theorem. [Akbari, Bidkhori, and Mohammadian, 2008]

$\mathbb{F}$ is arbitrary.

If $\Gamma(M_n(\mathbb{F}))$ is connected then $diam(\Gamma(M_n(\mathbb{F}))) \leq 6$.

Problem

$\mathbb{F} \neq \overline{\mathbb{F}}$, $\Gamma(M_n(\mathbb{F}))$ is connected. What values its diameter can achieve: 4, 5, 6 ?!

Example [Dolinar, Guterman, Kuzma, Oblak, 2014] The commuting graph for $M_9(\mathbb{Z}_2)$ is connected with diameter ≥ 5.

Theorem. [Shitov, 2015] There exist $\mathbb{F}$ and n such that $d(\Gamma(M_n(\mathbb{F}))) = 6$.

Theorem. [Shitov, 2015] $diam(\Gamma(M_n(\mathbb{R}))) = 4$.

Maps Preserving Matrix Invariants

Theorem. [Frobenius, 1896]

$T : M_n(\mathcal{C}) \to M_n(\mathcal{C})$ — linear, bijective

$$\det(T(A)) = \det A \qquad \forall A \in M_n(\mathcal{C})$$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathcal{C}), \det(PQ) = 1 :$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathcal{C})$$

or

$$T(A) = PA^tQ \quad \forall A \in M_n(\mathcal{C})$$

Theorem. [Dieudonné, 1949]

$\Omega_n(\mathbb{F})$ is the set of singular matrices

$T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ — linear, bijective,

$T(\Omega_n(\mathbb{F})) \subseteq \Omega_n(\mathbb{F})$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{F})$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{F})$$

or

$$T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{F})$$

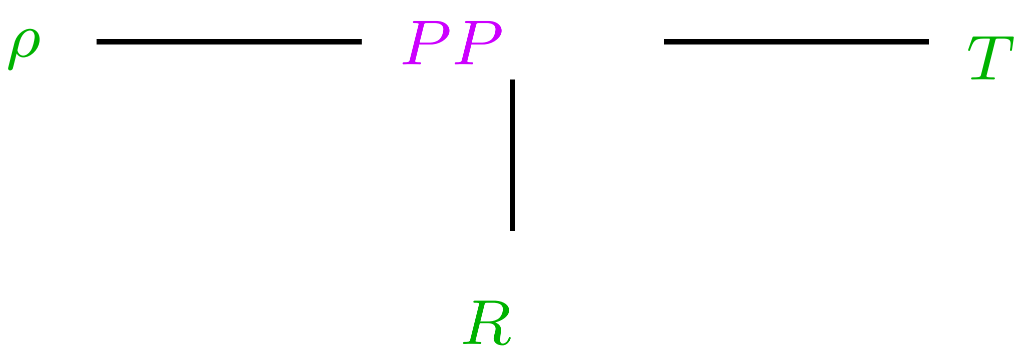
Preserve Problems

$\rho : M_n(R) \to S$ is a certain matrix invariant

$T : M_n(R) \to M_n(R)$

$$\rho(T(A)) = \rho(A) \quad \forall A \in M_n(R)$$

$T = ?$



Let $\mathbb{F}$ be a field

$\emptyset \neq S \subseteq M_n(\mathbb{F})$	$T(S) \subseteq S$
$\rho : M_n(\mathbb{F}) \to \mathbb{F}\ \forall A \in M_n(\mathbb{F})$	$\rho(T(A)) = \rho(A)$
$\sim: M_n(\mathbb{F})^2 \to \{0, 1\}$	$A \sim B \Rightarrow T(A) \sim T(B)$ $\forall A, B \in M_n(\mathbb{F})$
P – property in $M_n(\mathbb{F})$	$A \in P \Rightarrow T(A) \in P$

$$T = ?$$

The standard solution in linear case

There are $P, Q \in GL_n(\mathbb{F})$:

$$T(X) = PXQ \quad \forall X \in M_n(\mathbb{F})$$

or

$$T(X) = PXQ \quad \forall X^t \in M_n(\mathbb{F})$$

Preserver Problem

Theorem. [Watkins, 1976] $\overline{\mathbb{F}} = \mathbb{F}$, char$(\mathbb{F}) = 0$, $n \geq 4$, bijective linear $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves commutativity. Then Φ is of one of the following two standard forms: $\Phi(A) = cSAS^{-1} + f(A)I$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = cSA^tS^{-1} + f(A)I$ for all $A \in M_n(\mathbb{F})$, where $0 \neq c \in \mathbb{F}$, $S \in M_n(\mathbb{F})$ is invertible, and f is a linear functional on $M_n(\mathbb{F})$

[Omladič, Brešar, Šemrl, Fošner] — many results

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Theorem. [Dolinar, Guterman, Kuzma, Oblak, 2014]
$n \geq 3$, $|\mathbb{F}| \geq 2^{n-1}$. Bijective $\Phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ strongly preserves commutativity. Then $\exists$ homomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and $S \in GL_n$:

(i) $\Phi(A) = S\, p_A(A^\sigma) S^{-1}$ for all $A \in \mathfrak{D}_n(\mathbb{F}) \cup \mathcal{I}_n^1(\mathbb{F})$.

(ii) $\Phi(A) = S\, p_A(A^\sigma)^\top S^{-1}$ for all $A \in \mathfrak{D}_n(\mathbb{F}) \cup \mathcal{I}_n^1(\mathbb{F})$.

Here $p_A : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a matrix polynomial depending on A, $\mathfrak{D}_n(\mathbb{F}) \subset M_n(\mathbb{F})$ – diagonalizable matrices, $\mathcal{I}_n^1(\mathbb{F}) \subset M_n(\mathbb{F})$ – rank-one matrices.

Generalized centralizers

Definition Given a fixed $\omega \in \mathbb{F}$. We say that $A, B \in M_n(\mathbb{F})$ commute up to a factor ω if $AB = \omega BA$.

Definition $A, B \in M_n(\mathbb{F})$ is a quasi-commutative pair, if $\exists$ $0 \neq \varepsilon = \varepsilon(A, B) \in \mathbb{F}$ such that A, B commute up to a factor ε.

$A, B \in M_n(\mathbb{F})$ quasi-commute $\Longleftrightarrow$ $AB, \;\; BA$ are linearly dependent.

So, from linear algebra point of view quasi-commutativity relation is more natural than $AB = BA$.

• For given $\omega \in \mathbb{F}$, $\mathcal{X} \subseteq M_n(\mathbb{F})$ a generalized centralizer

$$\mathcal{C}^\omega(\mathcal{X}) = \bigcap_{A \in \mathcal{X}} \mathcal{C}^\omega(A),$$

where $\mathcal{C}^\omega(A) = \{B \in M_n(\mathbb{F}) : AB = \omega BA\}$.

• The quasi-centralizer is $\mathcal{C}^{\#}(\mathcal{X}) = \bigcap_{X \in \mathcal{X}} \bigcup_{\omega \in \mathbb{F}} \mathcal{C}^\omega(X) =$
$= \{B \in M_n(\mathbb{F}) | \forall X \in \mathcal{X} \ \exists \ 0 \neq \varepsilon_X \in \mathbb{F} : XB = \varepsilon_X BX\}$.

• $\mathcal{C}^\omega(\mathcal{C}^\omega(A)), A \in M_n(\mathbb{F})$, is a double generalized centralizer of A,

• $\mathcal{C}^{\omega^{-1}}(\mathcal{C}^\omega(A))$ is a symmetrized double generalized centralizer of A.

A well known Double Centralizer Lemma states that $\mathcal{C}(\mathcal{C}(A)) = \mathbb{F}[A]$ for an arbitrary $A \in M_n(\mathbb{F})$.

Example. If $A = I_n$, then $\mathcal{C}^\omega(A) = \{0\}$ for any $\omega \neq 1$. Therefore, $\mathcal{C}^\omega(\mathcal{C}^\omega(A)) = \mathcal{C}^{\omega^{-1}}(\mathcal{C}^\omega(A)) = M_n(\mathbb{F})$ and is not contained in $\mathbb{F}[A]$.

What can we obtain here?

Theorem. [Dolinar, Guterman, Kuzma, Markova]. Let $A \in M_n(\mathbb{F})$ be a nilpotent matrix with nilpotency index n_1, $\omega \in \mathbb{F} \setminus \{0, 1\}$.

(i) If $\exists k \leq n_1 : \omega^k = 1$ is primitive, then
$\mathcal{C}^\omega(\mathcal{C}^\omega(A)) = A^{k-1}\mathbb{F}[A^k]$, $\mathcal{C}^{\omega^{-1}}(\mathcal{C}^\omega(A)) = A\mathbb{F}[A^k]$.

(ii) If ω is not a primitive root of unity of degree $k \leq n_1$, then $\mathcal{C}^\omega(\mathcal{C}^\omega(A)) = \{0\}$ and $\mathcal{C}^{\omega^{-1}}(\mathcal{C}^\omega(A)) = \langle A \rangle$.

Invertible case

Theorem. [Dolinar, Guterman, Kuzma, Markova] Let $n \in \mathbb{N}$, $n \geq 2$, $\omega \in \mathbb{F} \setminus \{0, 1\}$, $\omega^{t+1} = 1$. Let $D = I_{n_1} \oplus \omega I_{n_2} \oplus \ldots \oplus \omega^t I_{n_{t+1}} \in M_n(\mathbb{F})$ be diagonal. Then $\mathcal{C}^{\omega^{-1}}(\mathcal{C}^{\omega}(D)) = \langle D \rangle$.

However, in general even the whole matrix algebra can lie in the second centralizer.

Lemma 1 [Dolinar, Guterman, Kuzma, Markova].
Let $\omega \neq 1$. Then $\mathcal{C}^{\omega}(A) = \{0\}$ if and only if $0 \notin \Big(\mathsf{Sp}(A) - \omega\,\mathsf{Sp}(A)\Big)$.

Let ω be fixed.

$A \prec B$ iff $\mathcal{C}^{\omega}(A) \subseteq \mathcal{C}^{\omega}(B)$.

$A \in M_n(\mathbb{F})$ is $\mathcal{C}^{\omega}$-minimal if $\mathcal{C}^{\omega}(A) \neq \{0\}$ and $\mathcal{C}^{\omega}(A) = \mathcal{C}^{\omega}(X)$ $\forall$ $X \in M_n(\mathbb{F})$ satisfying $\mathcal{C}^{\omega}(A) \supseteq \mathcal{C}^{\omega}(X) \supsetneqq \{0\}$.

$0 \neq A \in M_n(\mathbb{F})$ is $\mathcal{C}^{\omega}$-maximal if $\mathcal{C}^{\omega}(A) = \mathcal{C}^{\omega}(X)$ $\forall$ $X \in M_n(\mathbb{F})$ satisfying $\mathcal{C}^{\omega}(A) \subseteq \mathcal{C}^{\omega}(X) \subsetneqq M_n(\mathbb{F})$.

Theorem. [Dolinar, Guterman, Kuzma, Markova].

Let $\omega = -1$, $n \in \mathbb{N}, n \geq 2$, $\mathbb{F} = \overline{\mathbb{F}}$. Then $A \in M_n(\mathbb{F})$ is $\mathcal{C}^{-1}$-minimal $\iff$ A is similar to either

- $A' = 0 \oplus A_1 \in \mathbb{F} \oplus M_{n-1}(\mathbb{F})$, or
- $A' = J_{r_1}(\lambda) \oplus (-J_{r_2}(\lambda)) \oplus A_2$
  $\in M_{r_1}(\mathbb{F}) \oplus M_{r_2}(\mathbb{F}) \oplus M_{n-r_1-r_2}(\mathbb{F})$, $r_1 \geq r_2$, or
- $A' = J_r \oplus A_3 \in M_r(\mathbb{F}) \oplus M_{n-r}(\mathbb{F})$, $r \geq 2$,

where $\mathcal{C}^{-1}(A_i) = \{0\}$, $i = 1, 2, 3$, $\lambda \neq 0$, $\pm\lambda \notin \mathrm{Sp}(A_2)$.

Theorem. [Dolinar, Guterman, Kuzma, Markova].
Let $\omega \in \mathbb{F} \setminus \{-1, 0, 1\}$, $n \in \mathbb{N}, n \geq 2$, $\mathbb{F} = \overline{\mathbb{F}}$. Then $A \in M_n(\mathbb{F})$ is $\mathcal{C}^\omega$-minimal $\Leftrightarrow$ A is similar to

- $\lambda \oplus J_s(\omega\lambda) \oplus A_1 \in \mathbb{F} \oplus M_s(\mathbb{F}) \oplus M_{n-s-1}(\mathbb{F})$, or
- $(\omega\lambda) \oplus J_s(\lambda) \oplus A_2 \in \mathbb{F} \oplus M_s(\mathbb{F}) \oplus M_{n-s-1}(\mathbb{F})$, or
- $0 \oplus A_3 \in \mathbb{F} \oplus M_{n-1}(\mathbb{F})$,

where $\lambda \in \mathbb{F} \setminus \{0\}$ satisfies $\omega^{-1}\lambda, \lambda, \omega\lambda, \omega^2\lambda \notin \mathrm{Sp}(A_i)$, $s \in \{1, \ldots, n-1\}$, and each A_i satisfies $\mathcal{C}^\omega(A_i) = \{0\}$, $i = 1, 2, 3$.

Theorem. [Dolinar, Guterman, Kuzma, Markova].

Let $n \in \mathbb{N}$, $n \geq 2$, $\omega \in \mathbb{F} \setminus \{0, 1\}$. Let $\omega \in \mathbb{F}$, $\omega^k = 1$, ($k = \infty$ if ω is not a root of unity).

$A \in M_n(\mathbb{F})$ is $\mathcal{C}^\omega$-maximal $\Longleftrightarrow$ either

- k is arbitrary and A is nilpotent with nilindex $< k+2$,
- or $k \leq n$ and A is similar to a diagonal matrix $0_{n_0} \oplus \lambda I_{n_1} \oplus \omega\lambda I_{n_2} \oplus \ldots \oplus \omega^{k-1}\lambda I_{n_k}$, where $n_0 \in \mathbb{N} \cup \{0\}$, $n_i \in \mathbb{N}, i = 1, \ldots, k$, $\lambda \in \mathbb{F} \setminus \{0\}$.

Theorem. [Dolinar, Guterman, Kuzma, Markova]. Let $A \in M_n(\mathbb{F})$ be a nilpotent matrix with nilindex $n_1 \leq n$, $\omega \in \mathbb{F} \setminus \{0, 1\}$.

- If ω is not a root of unity of degree $k \leq n_1 - 2$, then A is $\mathcal{C}^\omega$-maximal.
- If ω is a primitive root of unity of degree k satisfying $k \leq n_1 - 2$, then A is not $\mathcal{C}^\omega$-maximal.

Theorem. [Dolinar, Guterman, Kuzma, Markova].
Let $\mathbb{F} = \overline{F}$, $\omega \in \mathbb{F} \setminus \{0, 1\}$. Then $A \in M_n(\mathbb{F})$ is $\mathcal{C}^{\omega}$-minimal and $\mathcal{C}^{\omega}$-maximal iff $\omega = -1$ and either

(i) $n = 2, 3$, A is similar to J_n, or

(ii) $n = 2$, A is similar to $\text{diag}\{\lambda, -\lambda\}$, $\lambda \neq 0$.

Theorem. [Dolinar, Guterman, Kuzma, Markova]

Let $\mathbb{F} = \overline{\mathbb{F}}$, $\omega \in \mathbb{F} \setminus \{0, 1\}$, $n \geq 3$. Then

(i) for $n \geq 4$ and for $n = 3$, $\omega \neq -1$, the set $\{X \in M_n(\mathbb{F}) | \mathcal{C}^\omega(X) \neq \emptyset, M_n(\mathbb{F})\}$ are partitioned into nonempty disjoint sets: (1) $\mathcal{C}^\omega$-minimal, (2) $\mathcal{C}^\omega$-maximal, and (3) $\mathcal{C}^\omega(X)$ is not extremal.

(ii) For $n = 3$, $\omega = -1$ the set $\{X \in M_n(\mathbb{F}) | \mathcal{C}^\omega(X) \neq \emptyset, M_n(\mathbb{F})\}$ is also partitioned into three nonempty sets (1), (2), (3), where (3) is disjoint with (1)$\cup$(2). The sets (1) and (2) intersect by matrices similar to J_3.

Problem. [Laffey] What is minimal dimension of a maximal commutative subalgebra of $M_n(\mathbb{F})$?

THANK YOU !