Maximal distance minimizers for a rectangle

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General problem

Problem

For a given compact set $M \subset R^2$ and $r > 0$ find a connected set Σ of minimal length such that

$$dist(\Sigma, M) \leq r,$$

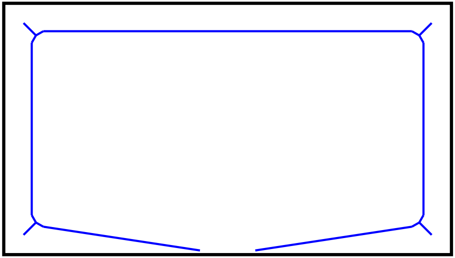
or, equivalently, $M \subset \overline{B_r(\Sigma)}$.

It is known that:

- a solution Σ exists;
- any solution Σ has no cycles.

Theorem

Let M be a rectangle, $0 < r < r_0(M)$. Then maximal distance minimizer is unique (up to symmetries of M). It is depicted below (the right part of the picture contains enlarged fragment of the minimizer; the labeled angle tends to $\frac{11\pi}{12}$ with $r \to 0$).



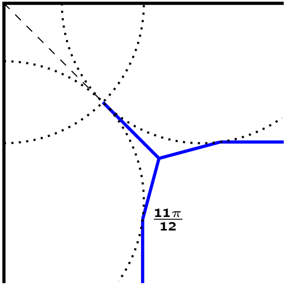


Figure: The minimizer for rectangle M and $r < r_0(M)$.

Steiner problem

In fact our problem can be considered as follows: to connect r-neighborhoods of all the points from M.

The Steiner problem is that of finding a set S with minimal length such that $S \cup A$ is connected, where A is a given planar compact subset.

It is known that

(i) a solution S exists;

(ii) S has no cycles;

(iii) $S \setminus A$ consists of line segments;

(iv) a point $x \in S \setminus A$ belongs to 1 or 3 line segments;

(v) the angle between two segments adjacent to the same vertex is greater or equal to $2\pi/3$.

A set S', which corresponds to the mentioned properties and connects A (but probably is too long) is called *local solution* for A.

Definition

A point $x \in \Sigma$ is called energetic, if for all $\rho > 0$ the set $\Sigma \setminus B_\rho(x)$ does not cover M i.e.

$$\mathrm{dist}(M, \Sigma \setminus B_\rho(x)) > r.$$

It is known that

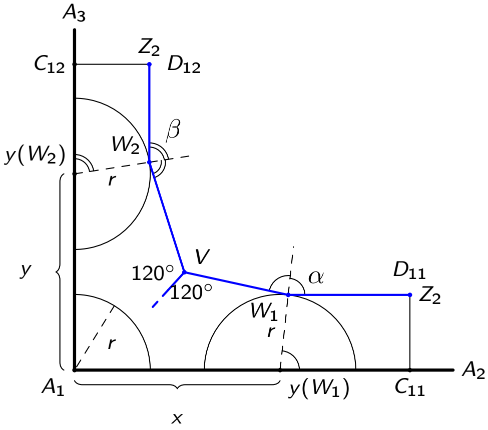
$$\Sigma = G_\Sigma \sqcup S_\Sigma,$$

where G_Σ is the set of all energetic points and S_Σ is Steiner part which means that Σ is a **local** solution of Steiner problem for G_Σ.

For every point $x \in G_\Sigma$ there exists a point $y(x) \in M$ (possibly non unique) such that $\mathrm{dist}(x, y(x)) = r$ and $B_r(y(x)) \cap \Sigma = \emptyset$.

First, we show that that $\Sigma \setminus B_{5r}(VERTICES)$ consists of 5 line segments.

Then we prove that the path between $\Sigma \cap \partial B_{5r}(A_1)$ contain two energetic points W_1, W_2 of degree 2 and one branching point V between W_1 and W_2.



Strategy of the proof II

Since W_1 and W_2 are energetic, $\partial B_r(y(W_1)) \cap \Sigma = \partial B_r(y(W_1)) \cap \Sigma = \emptyset$, so the neighborhoods of $y(W_1)$ and $y(W_2)$ should be covered by other points (say, Q_1 and Q_2). Also $\partial B_r(A_1)$ should contain a point from Σ, say, Q. Then

$$|\Sigma \cap ANGLE| \geq \mathrm{St}(Q_1, Q_2, Q, V) + |Z_2 W_2| + |W_2 V| + |V W_1| + |W_1 Z_1|.$$

Then we use computer calculations to show that x, y and α should be close to the corresponding parameters of the optimal Σ.

Finally we show that for such x, y and α there is no competitor corresponds to the following properties:

- $\mathrm{St}(Q_1, Q_2, Q, V)$ has the proper direction at V;
- the whole picture is optimal up to small changing of $y(W_1)$ and $y(W_2)$.

Theorem

For every closed convex curve M with minimal radius of curvature R and for every $r < R/5$ the set of minimizers contains only horseshoes.

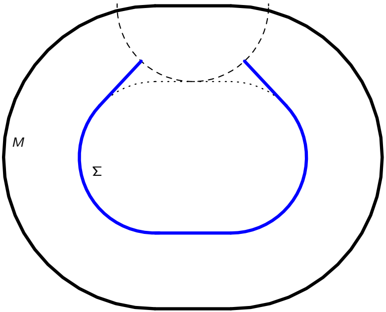


Figure: Horseshoe.

This theorem partially proves the conjecture Miranda, Paolini and Stepanov: for M being the circumference $\partial B(O)$ the set of minimizers contains only horseshoes.

Related results and open problems II

Define a *stadium* as the boundary of R-neighborhood of a segment. The following example shows that the previous theorem does not hold without the assumption on R/r.

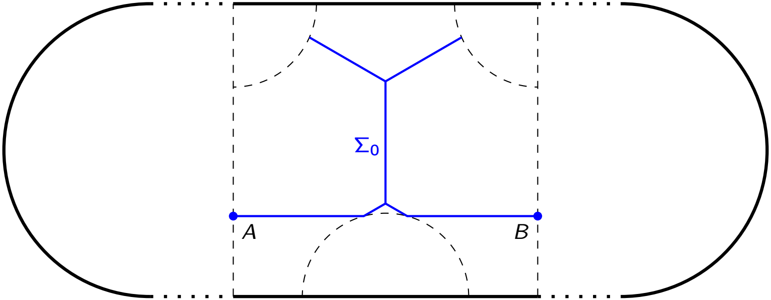


Figure: A horseshoe is not optimal for long enough stadium if $R < 1.75r$.

Sorry for your attention!