Multidimensional matrices in hypergraph matchings problems

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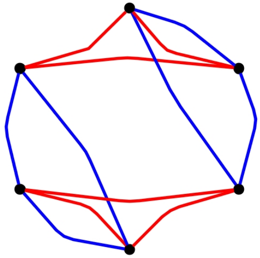
Hypergraphs

Hypergraph $H(X, W)$: X is the set vertices, W is the set of hyperedges.



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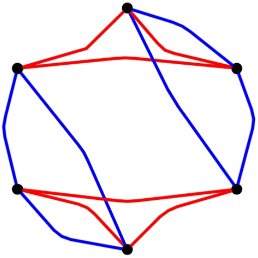

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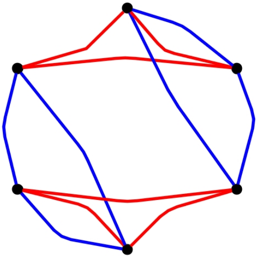

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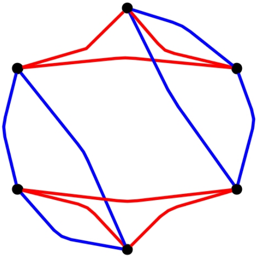

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A matching is a set $U \subset W$ such that each vertex is covered by at most one hyperedge $w \in U$.
A matching U is perfect if it covers all vertices of H.

Multidimensional matrices

d-dimensional matrix A of order n: an array $(a_\alpha)_{\alpha \in \mathcal{I}_n^d}$, $a_\alpha \in \mathbb{R}$ where, $\mathcal{I}_n^d = \{(\alpha_1, \ldots, \alpha_d) : \alpha_i \in [n]\}$.

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3-**dimensional** $(0,1)$-**matrix of order** 3

$$\left(\begin{array}{ccc|ccc|ccc} \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & \mathbf{0} & 1 & 1 & 0 & 0 & 0 & \mathbf{1} & 0 \end{array}\right)$$

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Partial diagonal $\mathcal{D}$ of length l: a set $\{\alpha^1, \dots, \alpha^l\}$ such that α_i, α_j are different at all positions.
Diagonal: a partial diagonal of the (maximal) length n.

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The permanent $\mathrm{per} A$ counts the number of unity diagonals in a $(0,1)$-matrix.

Adjacency matrices

An adjacency matrix of a d-graph $H(X, W)$ on n vertices is the d-dimensional $(0,1)$-matrix A of order n with $a_\alpha = 1$ if and only if $\alpha = (\alpha_1, \dots, \alpha_d)$ is a hyperedge in H.

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- $A(H)$ is symmetric: if α and β are permutably equivalent, then $a_\alpha = a_\beta$.

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A is a hypergraphic matrix if it is the adjacency matrix of some hypergraph.

d-adjacency matrices

A d-adjacency matrix $B(H)$ of a balanced d-partite d-graph H with parts $X_1, \dots, X_d$ of size n is a d-dimensional $(0,1)$-matrix of order n such that $b_\alpha = 1$ if and only if $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in X_i$ is a hyperedge of H.

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d-adjacency matrices B are blocks in the adjacency matrix A

$d = 2$

$$\begin{pmatrix} 0 & B'_{12} \\ B'_{21} & 0 \end{pmatrix}$$

$d = 3$

$$\left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & B'_{132} & 0 & B'_{123} & 0 \\ 0 & 0 & B'_{231} & 0 & 0 & 0 & B'_{213} & 0 & 0 \\ 0 & B'_{321} & 0 & B'_{312} & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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- A matching U of size l in $H \Leftrightarrow$ a positive partial diagonal $\mathcal{D}$ of length l in the d-adjacency matrix $B(H)$.
- The number of perfect matchings in H = the permanent of the d-adjacency matrix $B(H)$.

Degrees and planes

$H(X, W)$ is a d-graph on n vertices.

$deg(x)$ of a vertex x is the number of hyperedges containing x.
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A weight $w(\Gamma)$ of Γ is the sum of all entries in Γ.
A direction of a plane Γ is set of varying positions in Γ.

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$$deg(x) \cdot (d-1)! = w(\Gamma_x).$$

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Upper bounds on the number of perfect matchings

G is a simple graph, $\varphi(G)$ is the number of perfect matcings.

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Corollary

If r_i is the degree of a vertex x_i in G then

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If r_i is the degree of a vertex x_i, then

$$\varphi(H) \leq \left((d-1)!^n \mu^n \prod_{i=1}^{n} r_i \right)^{1/d}.$$

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Surveys:

1. V. Rödl, A. Ruciński. Dirac-type questions for hypergraphs – a survey. (2010).
2. Yi Zhao. Recent advances in Dirac-type problems for hypergraphs. (2016).

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Problem

Given n, k, d, l, find the smallest w such that every k-stochastic d-dimensional hypergraphic matrix of order n with k-planes of weight w has a positive symmetric partial diagonal of length dl.

Space barrier

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Weight of each line of M_{sb} is at least $n/d - 1$, M_{sb} has zero permanent.

Matrix $\mathcal{Z}_n^d$

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$\mathcal{Z}_n^d$ is equivalent to the block matrix with interlacing zero and unity blocks of (almost) equal sizes.

$$\mathcal{Z}_n^3 = \left(\begin{array}{cc|cc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} \end{array} \right)$$

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Proof for $d = 3$

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$2x = n/2 \Rightarrow n \equiv 0 \mod 4.$ Contradiction!

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The adjacency matrix of H_{db} is similar to the $\mathcal{Z}_n^d$.

Perfect matchings in d-partite d-graphs

Theorem (Aharoni, Georgakopoulos, Sprüssel, 2009)

Let H be a d-partite d-graph with parts $V_1, \ldots, V_d$ of sizes n. If for every legal $S \subset V \setminus V_1$, $|S| = d - 1$ we have $deg(S) > \frac{n}{2}$ and for every legal $S' \subset V \setminus V_2$, $|S'| = d - 1$ we have $deg(S') \geq \frac{n}{2}$, then H has a perfect matching.

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Equivalent theorem

Let A be a d-dimensional $(0, 1)$-matrix of order n. If for all lines Γ of some direction we have $w(\Gamma) > \frac{n}{2}$ and for all lines Γ' of other direction we have $w(\Gamma') \geq \frac{n}{2}$, then A has a positive permanent.

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The theorems are tight because of the matrix $\mathcal{Z}_n^d$.

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Conjecture (T., 2016)

Let A be a d-dimensional 1-stochastic $(0,1)$-matrix of order n.
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Let H be an n-balanced d-partite d-graph such that every legal $S \subset V$, $|S| = d - 1$ has the same degree.
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Vertex cover and Ryser's conjecture

A vertex cover in H is $U \subset V$ meeting each hyperedge at least once.
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D-extremal matrices of order 4

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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3-dimensional PD-extremal matrices of order 3

1. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

2. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

4. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

5. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 2/3 & 1/3 & 0 \\ 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix}.$$

6. $$\left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \Lambda = \begin{pmatrix} 3/4 & 1/2 & 0 \\ 1/2 & 1/4 & 0 \\ 1/2 & 1/4 & 0 \end{pmatrix}.$$

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The conjecture is true for some good infinite series of PD-extremal matrices and for all PD-extremal matrices of order $n = 2$.

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Thank you for your attention!