Induced and non-induced poset saturation problems

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Extremal problems vs Saturation problems

Triangle free graphs:

- Most number of edges: $\lfloor \frac{n^2}{4} \rfloor$ (Mantel 1908),
- Least number of edges in unextendable triangle-free graphs: $n-1$.

For graphs:

- Turán number: Erdős-Stone Simonovits theorem $\rightarrow$ $ex(n,F)=\Theta(n^2)$ unless F is bipartite. $ex(n,F)=O(n) \Longleftrightarrow F$ is a forest.
- $sat(n,G)$ = least number of edges in maximal/unextendable n-vertex G-free graphs $= O(n)$ Kászonyi, Tuza, 1986.

k-graphs:

- Turán number: ???
- $sat(n,H)=O(n^{k-1})$ Pikhurko, 1999.

Forbidden subposet problems

Theorem (Sperner, 1928)

If $\mathcal{F} \subseteq 2^{[n]}$ *does not contain* F, F' *with* $F \subsetneq F'$, *then* $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

Theorem (Erdős, 1945)

If $\mathcal{F} \subseteq 2^{[n]}$ *does not contain any* $(k+1)$*-chain* $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k+1}$, *then* $|\mathcal{F}| \le \sum_{i=1}^{k} \binom{n}{\lfloor (n-k)/2 \rfloor + i}$.

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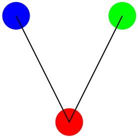
Theorem (Erdős, 1945)

If $\mathcal{F} \subseteq 2^{[n]}$ does not contain any $(k+1)$-chain $F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{k+1}$, then $|\mathcal{F}| \le \sum_{i=1}^{k} \binom{n}{\lfloor (n-k)/2 \rfloor + i}$.

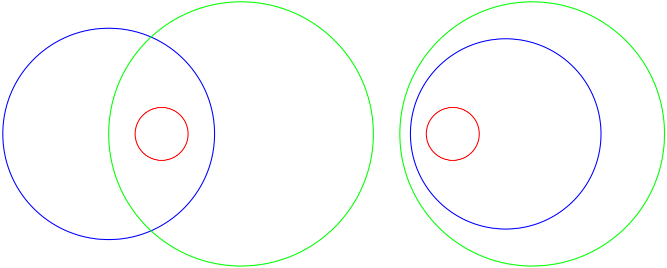
Katona and Tarján in 1983 introduced forbidden containment patterns described by posets.

Definition

Let P be a partially ordered set. We say that a family $\mathcal{F}$ of sets contains P if there exists an injection $i : P \to \mathcal{F}$ such that $p \leq_P q$ implies $i(p) \subset i(q)$.



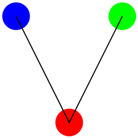
the poset $\vee$



Definition

Let P be a partially ordered set. We say that a subfamily $\mathcal{G} \subseteq \mathcal{F}$ of sets is

- a non-induced copy of P if there exists an injection $i : P \to \mathcal{G}$ such that $p \leq_P q$ implies $i(p) \subset i(q)$,
- an induced copy of P if there exists an injection $i : P \to \mathcal{G}$ such that $p \leq_P q$ if and only if $i(p) \subset i(q)$.



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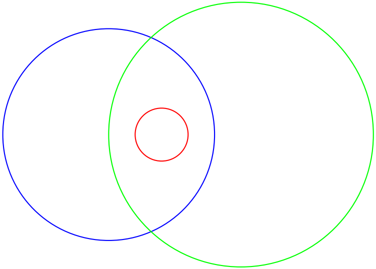

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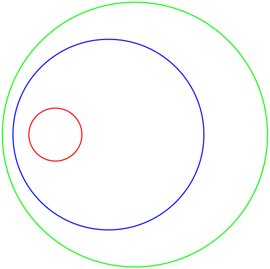

a non-induced copy of $\vee$

- If $\mathcal{F}$ does not contain a non-induced copy of P, then we say that $\mathcal{F}$ is P-free.
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$La(n, P)$ denotes the maximum size of a P-free family $\mathcal{F} \subseteq 2^{[n]}$.
$La^*(n, P)$ denotes the maximum size of an induced P-free family $\mathcal{F} \subseteq 2^{[n]}$.

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Erdős's theorem from 1945 about k-Sperner families states that

$$La(n, C_{k+1}) = La^*(n, C_{k+1}) = \sum_{i=1}^{k} \binom{n}{\lfloor (n-k)/2 \rfloor + i},$$

where C_{k+1} is the total ordering or chain on $k+1$ elements.

Erdős's result implies that $La(n, P) \leq (|P| - 1)\binom{n}{\lfloor n/2 \rfloor}$.

Methuku and Pálvölgyi (2017) proved $La^*(n, P) \leq C_P \binom{n}{\lfloor n/2 \rfloor}$ for all P.

Still unknown: do

$$\pi(P) = \lim_n \frac{La(n, P)}{\binom{n}{\lceil n/2 \rceil}} \qquad \pi^*(P) = \lim_n \frac{La^*(n, P)}{\binom{n}{\lceil n/2 \rceil}}$$

exist for all finite posets P?

Conjecture

- *For any poset P let $e(P)$ denote the most number of middle levels without creating a non-induced copy of P. Then $\pi(P)$ exists and is equal to $e(P)$.*
- *For any poset P let $e^*(P)$ denote the most number of middle levels without creating a induced copy of P. Then $\pi^*(P)$ exists and is equal to $e^*(P)$.*

Saturation forbidden subposet problems

$sat(n, P)$ = minimum size of a P-free $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F} \cup \{G\}$ contains a non-induced copy of P for any $G \in 2^{[n]} \setminus \mathcal{F}$,

$sat^*(n, P)$ = minimum size of an induced P-free $\mathcal{F} \subseteq 2^{[n]}$ such that $\mathcal{F} \cup \{G\}$ contains an induced copy of P for any $G \in 2^{[n]} \setminus \mathcal{F}$.

Saturation forbidden subposet problems - History

$G6$ =Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi, Patkós (2013)

Construction (G6)

For C_k: for $k \geq 3$, the family

$$\mathcal{F} = 2^{[k-3]} \cup \{[n] \setminus F : F \in 2^{[k-3]}\}$$

is C_k-saturating, so $sat(n, C_k) = sat^*(n, C_k) \leq 2^{k-2}$

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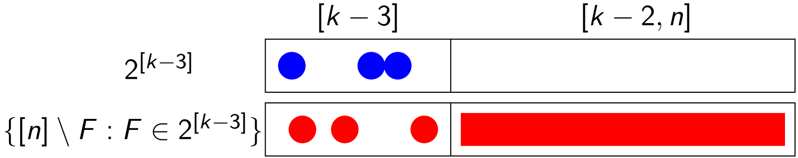

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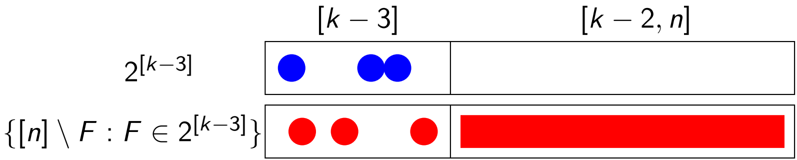

$\mathcal{F}$ is C_k-free as it is poset-isomorphic to $2^{[k-2]}$.

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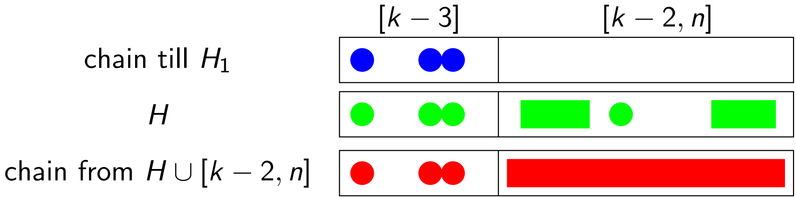
Adding a set $H = H_1 \cup H_2$ with $H_1 \subseteq [k-3]$ and $\emptyset \subsetneq H_2 \subsetneq [k-2, n]$ creates a k-chain:



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This is sharp if $k \leq 6$. On the other hand

Theorem (G6, 2013)

If $k \geq 7$, then $2^{\lfloor \frac{k-3}{2} \rfloor} \leq sat(n, C_k) \leq \frac{15}{16} 2^{k-2}$.

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Theorem (Morrison, Noel, Scott, 2014)

As k tends to infinity, we have $sat(n, C_k) \leq 2^{(0.98+o(1))k}$.

Saturation forbidden subposet problems - History II

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Found specific posets and classes of posets for which $sat^*(n, P) \to \infty$ as n tends to infinity.

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Ivan (2020+)
Linear lower bound on $sat^*(n, \bowtie)$ and $\sqrt{n} \leq sat^*(n, N)$.

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F7 & Martin, Smith, Walker & Ivan are “right” not to consider non-induced versions as:

Theorem (KLMPP, 2020+)

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Conjecture (KLMPP, 2020+)

For any poset P on k elements, we have $sat(n,P) \leq sat(n, C_k)$.

The proof

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GREEDY COLEX ALGORITHM

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GREEDY COLEX ALGORITHM

Greedy: consider sets of $2^{[n]}$ in some order $F_1, F_2, \ldots, F_{2^n}$. Let $\mathcal{F}_0 = \emptyset$.

$$\mathcal{F}_{i+1} = \begin{cases} \mathcal{F}_i \cup \{F_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\}, \text{does not contain any copy of} P \\ \mathcal{F}_i & \text{otherwise} \end{cases}$$

$\mathcal{F} := \mathcal{F}_{2^n}$ is clearly P-saturating.

The proof - II

Theorem (KLMPP, 2020+)

For any poset P, we have $sat(n, P) \leq 2^{|P|-2}$.

Colex: the co-lexicographic ordering of $Fin(\mathbb{Z}^+)$:
$A < B$ if and only if $\max(A \setminus B) \cup (B \setminus A)$ belongs to B.

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$\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{1, 4\} \ldots$

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The greedy colex algo is NOT what you would think!

The proof - III

Theorem (KLMPP, 2020+)

For any poset P, we have $sat(n,P) \leq 2^{|P|-2}$.

Let $F_1, F_2, \ldots, F_{2^{n-1}}$ be the enumeration of all sets in $2^{[n-1]}$ and let $G_i = [n] \setminus F_i$.

So the G_j's contain n, the F_i's do not.

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So the G_j's contain n, the F_i's do not.

The greedy colex algorithm considers the sets of $2^{[n]}$ in the order $F_1, G_1, F_2, G_2, \ldots, F_{2^{n-1}}, G_{2^{n-1}}$.

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$\emptyset, [n], \{1\}, [n] \setminus \{1\}, \{2\}, [n] \setminus \{2\}, \{1,2\}, [3,n], \dots, [n-1], \{n\}.$

$\mathcal{F}_{i+1} =$

$$\begin{cases} \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} \text{ is } P\text{-free} \\ \mathcal{F}_i \cup \{F_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\} \text{ is } P\text{-free, } \mathcal{F}_i \cup \{F_{i+1}, G_{i+1}\} \text{ not} \\ \mathcal{F}_i \cup \{G_{i+1}\} & \text{if } \mathcal{F}_i \cup \{F_{i+1}\} \text{ not } P\text{-free, } \mathcal{F}_i \cup \{G_{i+1}\} \text{ is } P\text{-free,} \\ \mathcal{F}_i & \text{otherwise.} \end{cases}$$

$\mathcal{F} := \mathcal{F}_{2^n-1}$ is the output of the greedy colex algorithm.

Theorem (KLMPP, 2020+)

For $1 \leq k \leq n$, let P be a k-element poset and let $\mathcal{F} := \mathcal{F}_{2^{n-1}}$ be the output of the greedy colex process. Then, $\mathcal{F}$ is P-saturating, $\mathcal{F} = \mathcal{F}_{2^{k-3}}$ and therefore $|\mathcal{F}| \leq 2^{k-2}$. In particular, $sat(n, P) \leq 2^{k-2}$ holds.

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Remark

Oh my God! Oh one God! OMG! (according to Dömötör: O1G!)

$$F_1, G_2, F_2, G_2, \ldots, F_{2^{k-3}}, G_{2^{k-3}}$$

is exactly the construction

$$2^{[k-3]} \cup \{[n] \setminus F : F \in 2^{[k-3]}\}$$

of the G6 guys!

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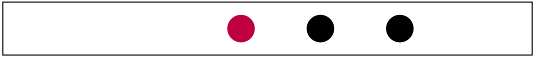
Can we say something about $\{\ell+2\}$ or $\{\ell+5, \ell+17\}$?



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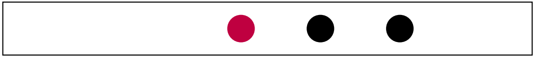
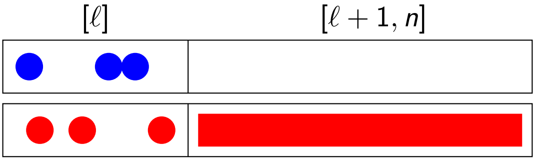


Sets considered so far:



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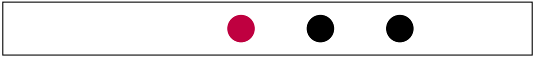

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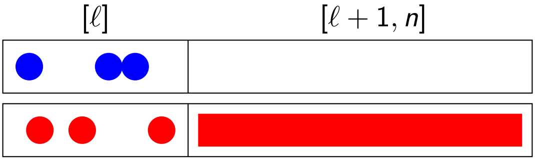

So if $\{\ell+1\}$ is not added, then later on the other two cannot be added either.

Induced results

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The following is implicitly in the work of F7

Lemma

For any poset P, the following are equivalent:

1. *There exists a constant C_P such that $sat^*(n,P) \leq C_P$ holds for all n.*
2. *There exists $x < y \leq m$ and a P-saturating $\mathcal{F} \in 2^{[m]}$ such that $\mathcal{F}$ does not separate x and y. (I.e. for all $F \in \mathcal{F}$ we have $|F \cap \{x,y\}| = 0, 2$.)*

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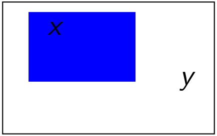

Consequences of the lemma:

Theorem

For any poset P,

- *either there exists a constant K_P with $sat^*(n,P) \leq K_P$*
- *or for all n, $sat^*(n,P) \geq \log_2 n$.*

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We conjecture the following strengthening.

Conjecture

For any poset P,

- *either there exists a constant K_P with $sat^*(n,P) \leq K_P$*
- *or for all n, $sat^*(n,P) \geq n+1$.*

Consequences of the lemma II:

Proposition

BoundedInducedSaturation is recursively enumerable.

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Is it recursive?

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- If it does, bad luck. :(

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- If it does, bad luck. :(

Yet, the greedy colex algo can be useful even if $sat^*(n, P) \to \infty$.

Let $\bowtie$ be the butterfly poset on four elements with $a, b < c, d$.

Analyzing the output of the greedy colex alg, one obtains

Theorem (KLMPP, 2020+)

$sat^*(n, \bowtie) \leq 6n - 10$.

Corollary

$sat^*(n, \bowtie) = \Theta(n)$.

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No :(For $2C_3$ the greedy colex process gives a quadratic family, but we can prove a linear upper bound.
Even worse: for $\Diamond'$ it gives an exponential family, while we can prove a linear bound, again.
But we do not have an example of a poset P with bounded saturation number, and the greedy colex giving an unbounded family.

More things we do not know

F7 introduced a class of posets for which $sat^*(n,P)$ is unbounded. We enlarged this class, while we showed sufficient conditions for $sat^*(n,P)$ to be bounded. But we do not understand what is happening and why.

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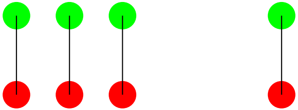
kC_2



Conjecture

Let k be a positive integer. There exists a constant c_k such that $sat^*(n, kC_2) \leq c_k$ if and only if k is odd.

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Odd values of k: we constructed families using circular intervals that are non-separating and conjectured they are saturating. The ones for $k = 3$ and 5 have this property, but the one for $k = 7$ does not work. However, the greedy colex does yield $sat^*(n, 7C_2) \leq 60$.

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For even values of k we were only able to prove the conjecture for $k = 2$.

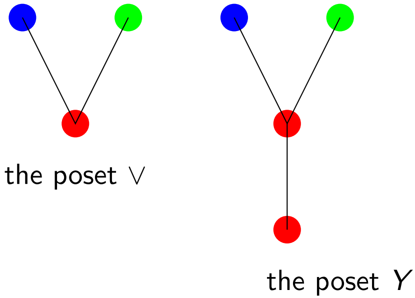
Theorem (KLMPP, 2020+)

If $\mathcal{F} \subseteq 2^{[n]}$ is saturating induced $2C_2$-free, then $\mathcal{F}$ contains a maximal chain in $[n]$. So $n + 1 \leq sat^*(n, 2C_2) \leq 2n$.

In a poset y covers x if there is no z with $x < z < y$.

A poset P is said to have UCTP (unique cover twin property) if whenever y covers x, then there is a z that is comparable with one of x and y and is incomparable to the other one.

That is either x is covered by not only y and thus the covering of x by y is not 'unique', or x is not the only one covered by x and thus x has a 'twin' covered by y.



Theorem (F7)

Let P be a poset that has UCTP. Then any P-saturating family is separating, thus $sat^*(n, P) \geq \log_2 n$.

A poset is called UCTP with top chain if it consists of two parts: a poset P_0 that has UCTP
and a chain such that every element of P_0 is smaller than every element of the chain.
For technical reasons, we also require $|P_0| \geq 2$ (i.e., the poset itself is not a chain).

Theorem (KLMPP, 20+)

Let P be a poset that has UCTP with top chain. Then any P-saturating family is separating, thus $sat^*(n, P) \geq \log_2 n$.

For example, the poset on four elements defined by $a < c; b < c; c < d$ (an upside-down 'Y') is a UCTP with top chain for which it was not known before whether it has an unbounded induced saturation function.

Recall: $e^*(P)$ is the most number of middle layers in $2^{[n]}$ without having an induced copy of P.

Theorem

If P is a poset with $e^*(P) \leq k-2$, then $sat^*(n, C_k + P) \leq K_P$ for some constant independent of n.

In particular, for $P = C_{i_1} + C_{i_2} + \cdots + C_{i_k}$ with $i_1 \geq \max\{i_j : 2 \leq j \leq k\} + 2$, then $sat^*(n, P) \leq K_P$ for some constant K_P independent of n.

Thank you for your attention!